1990 ALGEBRA PRELIMINARY EXAMINATION

The exam is divided into three sections: Group Theory, Ring Theory and Galois Theory. You have the option of selecting nine of the twelve problems as long as you select at least three from each section. If time permits, you are encouraged to work all twelve problems.

Group Theory

1. (a) Prove that any subgroup of a cyclic group is itself cyclic.
(b) Let N be a normal subgroup of an arbitrary group G and suppose that G/N is cyclic. Prove that if H is any subgroup of G, then $H/H \cap N$ is also cyclic.
(c) If N is a normal cyclic subgroup of a group G, prove that any subgroup of N is also normal in G.
(d) Give the simplest/smallest example possible of a group G that has a normal subgroup of a normal subgroup that is not normal in G.

2. (a) Outline a proof that A_n is simple if $n \geq 5$.
(b) Show how (a) implies that S_n is not solvable when $n \geq 5$.

3. (a) Prove that every nontrivial group of prime power order p^n has a nontrivial center.
(b) Sketch a proof of the fact that any two p-Sylow subgroups of a finite group G are conjugate in G; let $|G| = p^n t$ where $n \geq 1$ and $(p, t) = 1$, with p of course a prime.

4. Prove that any finitely generated abelian group is a direct sum of cyclic groups. (If you do Problem 4 of the Ring Theory Section, which is the same problem for modules over PID's instead of $\mathbf{Z}$, you will receive credit for *both* problems *without* doing the special case called for in this problem.)

Ring Theory

1. As usual, let $\mathbf{R}$ and $\mathbf{C}$, respectively, denote the fields of real and complex numbers.
   (a) Prove that $\mathbf{R}$ and $\mathbf{C}$ have the same additive structure,
   $$(\mathbf{R}, +) \cong (\mathbf{C}, +).$$
   (b) Prove that $\mathbf{R}$ and $\mathbf{C}$ do not have the same multiplicative structure,
   $$(\mathbf{R} \setminus \{0\}, \cdot) \ncong (\mathbf{C} \setminus \{0\}, \cdot).$$

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2. Let F be a field and let $F[x, y] = (F[x])[y]$ be the polynomial ring in two variables over F.
   (a) Prove that $F[x, y]$ is not a PID.
   (b) Prove that $F[x, y]$ is a UFD.

3. In this problem, assume that R is a ring with 1 and that all modules are left unitary R-modules.
   (a) Let
   $$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$
   be an exact sequence of modules. Prove that if there exists a homomorphism $\gamma : C \to B$ such that $\beta\gamma = 1_C$, then $B \cong A \oplus C$.
   (b) Prove that any projective module is a direct summand of a free module.
   (c) Give an example of a projective module that is not free.

4. Prove that any finitely generated module over a PID is a direct sum of cyclic modules. (Compare with Problem 4 of the Group Theory Section.)

Galois Theory

The notation for this section will be as follows: $K \subseteq F$ is a field extension with Galois group $\mathrm{Aut}_K F$. If $G \leq \mathrm{Aut}_K F$, G' denotes the fixed field of G, and if L is an intermediate field of $K \subseteq F$, then $L' = \mathrm{Aut}_L F$.

1. (a) If $K \subseteq F$ is finite-dimensional, prove that $K \subseteq F$ is algebraic.
   (b) Give an example of an algebraic extension that is not finite and explain why your example works.

2. (a) Find the Galois group for each polynomial over $\mathbf{Q}$ and briefly explain why your answers are correct.
   $$x^5 - 1 \qquad x^5 - 5x^3 - 20x + 5 \qquad x^4 - x^2 - 2$$
   (b) Which if any of the polynomials in part (a) is *not* solvable by radicals over $\mathbf{Q}$?

3. Recall that if L and M are subfields of F, LM denotes the smallest subfield of F that contains both L and M. If L and M are intermediate fields of a finite Galois extension $K \subseteq F$, prove that $(LM)' = L' \cap M'$. (Hint: One inclusion is clear. Use the Fundamental Theorem of Galois Theory to verify the reverse inclusion.)

4. Suppose $K \subseteq F$ is a normal field extension and E is an intermediate field.
   (a) Prove: E is normal over K if and only if E is stable. (Recall that E is *stable* if $\sigma(E) \subseteq E$ for all $\sigma \in \mathrm{Aut}_K F$.)
   (b) Prove: If E is normal over K, then $K'/E' \cong \mathrm{Aut}_K E$.