1992 ALGEBRA PRELIMINARY EXAMINATION

This exam is divided into three sections: Group Theory, Ring Theory and Galois Theory. In each section you are required to do problem 1 and all but one of those that remain.

Group Theory

1. State the Krull-Schmidt Theorem.

2. Give a detailed outline of the proof of the Fundamental Theorem for finitely generated abelian groups.

3. Let G be a finite proup of order pq, where p and q are primes. Prove:
   (a) If $p = q$, show that G is abelian. Must G be cyclic?
   (b) If $p \leq q$ and $p \nmid (q-1)$, show that G is cyclic.

4. If a group G contains a proper subgroup of finite index, show that G contains a proper normal subgroup of finite index.

5. For a group G, let $C(G)$ denote the center of G. Set $C_1(G) = C(G)$ and for $n \geq 2$ define $C_n(G)$ to be the subgroup of G such that $C_n(G)/C_{n-1}(G) = C(G/C_{n-1}(G))$. Recall that G is *nilpotent* if $C_n(G) = G$ for some n. If p isa prime, show that every finite p-group is nilpotent.

Ring Theory

In this section, R is a ring with 1, and all modules are unitary left R-modules.

1. State the Artin-Wedderburn Theorem.

2. Show that free modules are projective. Conclude that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

   is an exact sequence of vector spaces over a field, then $B \cong A \oplus C$.

3. Let A and B be modules. If A is simple, prove each of the following statements.
   (a) Every nonzero homomorphism $f : A \to B$ is a monomorphism.
   (b) Every nonzero homomorphism $g : B \to A$ is an epimorphism.
   (c) $\mathrm{Hom}(A, A)$ is a division ring.

4. Let $R = \mathrm{Mat}_n(D)$, the ring of all $n \times n$ matrices over a division ring D. Show directly that R is simple and describe the center $Z(R)$ of R.

Galois Theory

1. State the Fundamental Theorem of Galois Theory.

2. Compute the Galois group of $f(x) = x^3 - 5 \in \mathbf{Q}[x]$ and determine a splitting field for f over $\mathbf{Q}$.

3. Find a splitting field F for $f(x) = x^4 + x^3 + x^2 + x + 1$ over $\mathbf{Q}$ and list all intermediate fields between $\mathbf{Q}$ and F.

4. Show that if F is a finite field, then $|F| = p^n$ for some prime p and integer $n \geq 1$.