# ALGEBRA PRELIMINARY EXAM

May 9, 1998

## I. GROUP THEORY

Do problem 1 and any four of problems 2 through 6.

1. Prove the First Sylow Theorem: If $G$ is a finite group of order $p^n m$, where $n$ is a positive integer and $p$ is a prime not dividing $m$, then $G$ has a subgroup of order $p^n$.

2. Construct a list of abelian groups of order 1998 such that every abelian group of order 1998 is isomorphic to exactly one group on the list.

3. Show that, up to isomorphism, there is only one group of order 33.

4. Suppose $G$ is a group of order $p^n$ for some prime $p$ and positive integer $n$.

   a. Show that $G$ has a nontrivial center.

   b. Prove that if $|G| = p^2$, then $G$ is abelian.

   c. Give an example of a prime $p$ and a group $G$ of order $p^3$ that is *not* abelian.

5. Prove or give a counterexample: Every solvable group is nilpotent.

6. Suppose $n \geq 5$ and $G$ is a simple group of order $n!/2$. Prove that $G \cong A_n$ if and only if $G$ has a subgroup of index $n$.

## II. RING THEORY

Do problem 7 and any four of problems 8 through 12.

7. Prove that every Euclidean Integral Domain (EID) is a Principal Ideal Domain (PID).

8. Let $R$ be a commutative ring with identity and let $S$ be a multiplicative subset of $R$.

   a. Give a careful definition of $S^{-1}R$, the ring of quotients of $R$ by $S$, and show that its addition is well-defined.

   b. Let $R = \mathbb{Z}$ and $S = \{n \in \mathbb{Z} \,|\, 5 \nmid n\}$. Prove or disprove: There exist ring epimorphisms $\varphi : S^{-1}R \to F_1$ and $\psi : S^{-1}R \to F_2$, with $F_1$ and $F_2$ nonisomorphic fields.

9. Let $R$ be a ring with identity and let $X$ be a nonempty set. Prove that there exists an object free on $X$ (relative to the forgetful functor, as usual) in the category of unitary left $R$-modules.

10. Let $R$ be a ring and let $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ be an exact sequence of left $R$-modules. Prove that there exists an $R$-homomorphism $h : A_2 \to B$ such that $gh = 1_{A_2}$ if and only if the given sequence is isomorphic to $0 \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to 0$, where $\iota_1(a_1) = (a_1, 0)$ and $\pi_2((a_1, a_2)) = a_2$ $(a_i \in A_i)$.

11. Let $R$ be a PID. Prove that a left $R$-module $A$ is injective if and only if it is divisible (meaning, for each $a \in A$ and $0 \neq r \in R$, there exists $b \in A$ such that $rb = a$).

12. Let $R$ be a ring. Let $A$, $A'$ be right $R$-modules, let $B$, $B'$ be left $R$-modules, and let $f : A \to A'$, $g : B \to B'$ be $R$-homomorphisms.

    a. Prove that there exists a unique group homomorphism $f \otimes g : A \otimes_R B \to A' \otimes_R B'$ satisfying $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ for all $a \in A$, $b \in B$.

    b. Prove or give a counterexample: If $f$ is injective, then so is $f \otimes 1_B : A \otimes_R B \to A' \otimes_R B$.