ALGEBRA PRELIM

JUNE 5, 1999

1. Group Theory

Select any **three** numbered problems from this section to work.

1. Let G be a finite group.
   (a) Show if G is Abelian, then for each positive integer m dividing $|G|$, there exists a subgroup of G of order m. (Hint: use the Fundamental Theorem)
   (b) Provide a counter-example to (a) if G is not assumed to be Abelian.

2. Let G be a finite group, and recall that G acts on itself by conjugation.
   (a) Given $x \in G$, show that the number of elements in the *conjugacy class* of x is $[G : C_G(x)]$, where $C_G(x)$ is the *centralizer* of x in G.
   (b) If $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are the distinct conjugacy classes of G, derive the formula

$$|G| = \Sigma_{j=1}^{n}[G : C_G(x_j)].$$

3. State the three Sylow Theorems and use them to show that every group of order 21 is not simple.

4. State Cauchy's Theorem and use it to show: For a fixed prime p and a finite group G, every element of G has order p^k for some $k \geq 0$ if and only if $|G| = p^m$ for some $m \geq 0$.

2. Ring Theory

Select any **four** numbered problems from this section to work.

1. Prove that any finitely generated module over a principal ideal domain is a direct sum of cyclic modules.

2. Prove that a torsion-free module over an integral domain is divisible if and only if it is injective.
3. Prove Hilbert's Basis Theorem: If R is a commutative Noetherian ring, then so is the polynomial ring $R[x]$.
4. Using the definition that a Dedekind domain is an integral domain in which every nonzero ideal is invertible, show that in a Dedekind domain R, every proper ideal is a product of (one or more) prime ideals.
5. Let R be a ring with 1.
   (a) Define the Jacobson radical of R, hereafter denoted by $J(R)$.
   (b) Prove Nakayama's Lemma: If M is a finitely generated module such that $J(R)M = M$, then $M = 0$.