ALGEBRA PRELIMINARY EXAM

Attempt at least **3** problems from each section!

1. Group Theory

1. State the three Sylow Theorems and give an outline of their proofs.
2. State and prove the Frattini Argument.
3. Let G be a finite group, and suppose that p is the smallest prime dividing $|G|$. Show that every subgroup U of G with $|G : U| = p$ is normal in G.
4. Let p and q be primes. Show that every group G with $|G| = pq$ is solvable.
5. Show that a finite group G is nilpotent if and only if $|Syl_p(G)| = 1$ for all primes p.

2. Rings and Modules

1. Let R be a principal ideal domain. Show that a submodule of a free module is free.
2. Show that every integral domain R has a field of quotients $Q(R)$ which is unique up to isomorphism.
3. State the structure theorem of finitely generated unitary modules over a PID.
4. Give an example for each of the following cases. Explain why your examples work.
   a) A UFD that is not a Noetherian domain.
   b) A Noetherian domain that is not a UFD.
5. Let R be a ring. Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

   is a commutative diagram of R-module homomorphisms such that each row is exact. Show
   a) If α and γ are monomorphisms, then β is a monomorphism.
   b) If α and γ are epimorphisms, then β is an epimorphism.
   c) If α and γ are isomorphisms, then β is an isomorphism.

3. Fields and Galois Theory

1. Let F be an extension field of a field K, and $u \in F$ algebraic over K. Show that the rings $K(u) = K[u] \simeq K[x]/(p(x))$, where $p(x)$ is the irreducible polynomial of u over K.
2. Suppose the fields K, E, F satisfy that $K \subset E \subset F$, and F is a finite dimensional extension over K. Prove that $[F : K] = [F : E][E : K]$.
3. Determine the Galois groups of the following polynomials over $\mathbb{Q}$:
   a) $f(x) = (x^2 + 1)(x^2 + 2)$
   b) $f(x) = x^3 - 3x + 1$
   c) $f(x) = x^{2008} - 1$
4. Let F be a finite field with 3^{12} elements; and view F as an extension field of $\mathbb{Z}_3$.
   a) Determine the Galois group of F over $\mathbb{Z}_3$.
   b) Draw the intermediate field diagram between F and $\mathbb{Z}_3$, and the subgroup diagram of $\mathrm{Aut}_{\mathbb{Z}_3} F$.
5. Let $K \subset F \subset \overline{K}$ be fields such that $\overline{K}$ is an algebraic closure of K, and F is a finite dimensional extension field of K. Let $\{F : K\}$ denote the number of distinct K-isomorphisms from F to certain extension field of K in $\overline{K}$. Prove that $|\mathrm{Aut}_K F| \leq \{F : K\} \leq [F : K]$.