ALGEBRA PRELIMINARY EXAMINATION

June 13, 2009

I. Groups

Do problems 1 and 2 and any three of the remaining five.

1. State and prove the Second Isomorphism Theorem (also known as the Diamond Theorem).

2. (a) Define *solvable group* (and include the definitions of any nonelementary terms you use).
   (b) Let p and q be prime numbers. Prove that a group of order pq is solvable. (Hint: Reduce to the case $p \neq q$ and then use Sylow theory.)

3. Let G be a group and let H be a subgroup of G. Prove that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the group $\mathrm{Aut}(H) \leq \mathrm{Sym}(H)$ of all automorphisms of H. ($N_G(H)$ denotes the normalizer of H in G and $C_G(H)$ denotes the centralizer of H in G.)

4. (a) Let m and n be positive integers with $\gcd(m, n) = 1$. Prove that
$$\mathbf{Z}_m \oplus \mathbf{Z}_n \cong \mathbf{Z}_{mn}.$$
   (b) Determine the number of isomorphism classes of abelian groups of order 400.

5. (a) Define *composition series* of a group G (and include the definitions of any nonelementary terms you use).
   (b) Prove or disprove: Every group has a composition series.
   (c) Find the composition factors of the group $G = S_8 \times \mathbf{Z}_{12} \times D_{10}$, where D_{10} is the dihedral group of order 10 (no proof required).

6. Let G_1 and G_2 be groups.
   (a) Prove that there exists a product of the family $\{G_i\}$ in the category of groups.
   (b) Let $\iota_i : G_i \to G_1 \times G_2$ be the injections defined by $\iota_1(g_1) = (g_1, e_2)$ and $\iota_2(g_2) = (e_1, g_2)$, where e_i is the identity element of G_i. Prove that $(G_1 \times G_2, \{\iota_i\})$ need not be a coproduct of the family $\{G_i\}$ in the category of groups.

7. Give a Venn diagram representation of the following classes of finite groups: abelian, p-group, nilpotent, solvable, simple, cyclic. Provide an example of a group in each cell of the diagram. For the example you give to show that solvable groups and nilpotent groups are not the same, give a justification.

II. Rings, Modules, and Galois Theory

Do problems 8 and 9 and any three of the remaining five.

8. Suppose that R is a PID and that $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$ is an ascending chain of ideals of R indexed by the positive integers. Show that there is a positive integer k such that $I_n = I_k$ for all $n \geq k$.

9. Let A be a subgroup of the additive group of rationals $\mathbb{Q}$ and assume that $1 \in A$. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} A \cong \mathbb{Q}$ as abelian groups.

10. If p is a prime number, show that $\mathbb{Z}_p[x]$ contains an irreducible polynomial of degree n for every positive integer n.

11. Determine the Galois group of each of the following polynomials over $\mathbb{Q}$, and then use the Fundamental Theorem to list all the intermediate fields between $\mathbb{Q}$ and the splitting field $F \subseteq \mathbb{C}$. (By determining the Galois group, we mean finding a "familiar" group to which the Galois group is isomorphic.)

    (a) $x^4 - 5x^2 + 6$.

    (b) $x^3 - 2$.

12. If R is a commutative ring with identity, prove that $a \in R$ is nilpotent if and only if a is contained in every prime ideal of R.

13. Suppose that $K \subseteq F_1$ and $K \subseteq F_2$ are field extensions (of the same field K) with respective Galois groups G_1 and G_2. If there is a K-isomorphism $f : F_1 \to F_2$, prove that $G_1 \cong G_2$.

14. Let M be a (left) R-module over some ring R, and suppose that N is a submodule of M. Prove that $M = N \oplus K$ for some submodule K if and only if there is an R-module homomorphism $f : M \to N$ that restricts to the identity on N.