ALGEBRA PRELIMINARY EXAM

Attempt at least **3** problems from each section!

1. Group Theory

(1) Let $\{A_i \mid i \in I\}$ be a family of abelian groups. Prove that the product and the coproduct of $\{A_i \mid i \in I\}$ exist in the category of abelian groups.
(2) Let A be a finitely generated abelian group. If $A \times A \simeq B \times B$ as groups, prove that $A \simeq B$.
(3) Prove the Cayley's Theorem: Every finite group G is isomorphic to a subgroup of S_n with $n = |G|$.
(4) Let p and q be two primes with $3 \leq p < q$. Prove that a group of order p^2q^2 has a normal subgroup of order q^2. *(Hint: Use Sylow theory)*
(5) Prove that every nilpotent group is solvable.

2. Rings and Modules

(1) Show that $\mathbb{Z}[x]$ is not a principal ideal domain (PID), but $\mathbb{Z}[x]$ is a unique factorization domain (UFD).
(2) State and prove Eisenstein's Criterion.
(3) State and prove the Hilbert Basis Theorem.
(4) Show: Let R be an integral domain. If M is a finitely generated R-module, then M/M_t is isomorphic to a submodule of a finitely generated free module.
(5) State the fundamental theorem for finitely generated modules over a PID.

3. Fields and Galois Theory

(1) Show: If $F \geq K$ is algebraic, and R is a ring with $F \geq R \geq K$, then R is a field.
(2) State the Fundamental Theorem of Galois Theory.
(3) (a) Show that the field of complex numbers is Galois over $\mathbb{Q}$.
    (b) Determine the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})$.
(4) (a) Prove that if $\mathbf{k}$ is algebraically closed, then $\mathbf{k}$ is not a finite field.
    (b) Let $\mathbf{k}$ be a field and f be a polynomial over $\mathbf{k}$ with $\deg f = n$. If $\mathbf{L}$ is a splitting of f over $\mathbf{k}$, then argue that $[\mathbf{L} : \mathbf{k}] \mid n!$.
(5) Let K be a field, and E its algebraic closure. Show that E is countable and infinite if K is finite, and $|E| = |K|$ if K is infinite.