NAME: Daniel James ALGEBRA PRELIMINARY EXAM
December 3, 2013

Make Sure You Answer at Least 2 Questions From Each Group

Extra Answers are Synonymous with Extra Credit

1 Group Theory

1. Let G be a finite group. State the three Sylow Theorems. Do (a) or (b) below:

   (a) Show that no group of order 28 can be simple.

   (b) Let G be a group of order 168. Determine the number of elements of order 7.

2. Prove the Frattini Argument; If G is a finite group, and H is normal in G, then for $P \in Syl_P(H)$, we have $G = HN_G(P)$.

3. Give a statement of the Class Equation.

2 Ring Theory

1. Let R be a commutative, noetherian domain. Show that any nonunit is a finite product of irreducible elements. Conclude that a commutative

noetherian domain such that irreducible elements are prime, is a UFD.

2. State the Artin-Wedderburn Theorem. As an example, show that the vectors

$$X = \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_n \end{pmatrix},$$

form a simple left $R = Mat_{nxn}(D)$ module where D is a division ring, and $d_1, d_2, \dots, d_n \in D$.

3. Let R be a commutative, noetherian domain. Give at least 3 different, but equivalent characterizations for R to be a Dedekind domain (1 can be the definition).

3 Field Theory

1. (a) State the Fundamental Theorem of Galois.

   (b) Assuming the Statement of 2. is correct, how many proper subfields between $F = \mathbb{Q}[\sqrt[3]{2}, e^{\frac{2i\pi}{3}}]$ and $\mathbb{Q}$ are there?

2. The splitting field of $f(x) = x^3 - 2$ is $F = \mathbb{Q}[\sqrt[3]{2}, e^{\frac{2i\pi}{3}}] = \mathbb{Q}[\sqrt[3]{2}, \sqrt{-3}]$. Show that the Galois group of F over $\mathbb{Q}$ is the symmetric group S_3.

3. Argue that the splitting field of $h(x) = x^p - 1$ is $\mathbb{Q}[e^{\frac{2i\pi}{p}}]$.