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Preliminary Doctoral Examination in Analysis

Name: ________________________________

Notes. You have four hours to complete this exam. You may not use any books, notes, or electronic communication devices. Problems 1–5 are mandatory. Problem 6 is a bonus problem, good for extra credit.

1. Let X be a nonempty set, $\mathcal{M}$ a nonempty collection of subsets of X, and μ a mapping from $\mathcal{M}$ into $[0,\infty]$.

(a) What does it mean to say that $\mathcal{M}$ is an *algebra*? Assuming that $\mathcal{M}$ is an algebra, what does it mean to say that μ is *finitely additive*? Still assuming that $\mathcal{M}$ is an algebra, what does it mean to say that μ is a *pre-measure*?

(b) What does it mean to say that $\mathcal{M}$ is a σ*-algebra*? Assuming that $\mathcal{M}$ is a σ-algebra, what does it mean to say that μ is *countably additive*, *continuous from below*, or *continuous from above*, respectively? Still assuming that $\mathcal{M}$ is a σ-algebra, what does it mean to say that μ is a *measure*? Assuming that $\mathcal{M}$ is a σ-algebra and μ is a measure, what does it mean to say that μ is σ-finite?

(c) Suppose that $\mathcal{M}$ is a σ-algebra and that μ is finitely additive and satisfies $\mu(\emptyset)=0$. Prove that μ is countably additive if and only if μ is continuous from below.

2. Let $(X,\mathcal{M},\mu)$ be a measure space.

(a) State the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem (complete with all assumptions and assertions).

(b) Use the theorems in Part (a) to prove the "series version" of the Dominated Convergence Theorem: Suppose $(f_k)_{k\in\mathbb{N}}$ is a sequence of measurable numerical functions on X such that $\sum_{k\in\mathbb{N}}\int_X |f_k|\,d\mu<\infty$. Then f_k is integrable for every $k\in\mathbb{N}$, $\sum_{k\in\mathbb{N}} f_k$ converges pointwise a.e., the sum is integrable, and $\int_X\big(\sum_{k\in\mathbb{N}} f_k\big)d\mu=\sum_{k\in\mathbb{N}}\big(\int_X f_k\,d\mu\big)$.

3. Let $(X,\mathcal{M},\mu)$ be a measure space, $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $p,q\in[1,\infty)$.

(a) Give the definition of the space $L^p=L^p(X,\mathcal{M},\mu,\mathbb{K})$ and its norm, $\|\cdot\|_p$. (Distinguish between "measurable functions" and "equivalence classes of measurable functions.")

(b) Suppose that $\mu(X)<\infty$ and $p<q$. Show that $L^q\subset L^p$ and $\|f\|_p\le\mu(X)^{1/p-1/q}\|f\|_q$ for all $f\in L^q$. (*Hint*: Hölder's inequality!) Conclude that convergence in L^q implies convergence in L^p. Give examples showing that, in general, without the assumption $\mu(X)<\infty$, neither $L^q\subset L^p$ nor $L^p\subset L^q$.

4. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let f be a numerical function on $X \times Y$, measurable with respect to the product σ-algebra $\mathcal{M} \times \mathcal{N}$.

(a) State the theorems of Tonelli and Fubini (complete with all assertions, under the above assumptions).

(b) Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := e^{-xy} \sin x$. Given $b \in (0, \infty)$, show that f is Lebesgue-integrable on $(0, b) \times (0, \infty)$ and compute the two iterated integrals. Use the result to show that $\int_0^\infty \frac{\sin x}{x} dx$ exists as an improper Riemann integral and determine its value. Carefully justify your conclusions!

Hint. For every $y \in \mathbb{R}$, an antiderivative for $x \mapsto e^{-xy} \sin x$ is given by $x \mapsto -\frac{e^{-xy}}{1+y^2} (y \sin x + \cos x)$.

5. Let $(X, \mathcal{M})$ be a measurable space and let μ and ν be measures on $\mathcal{M}$.

(a) Define what it means to say that ν is *absolutely continuous* with respect to μ. Further, define what it means to say that ν has a *density* with respect to μ. State the Radon-Nikodym Theorem (under the above assumptions).

(b) Show that if ν is finite, and if f and g are densities of ν with respect to μ, then $f = g$ a.e. with respect to μ.

(c) Suppose that μ is finite and let f be a nonnegative $\mathcal{M}$-measurable function on X. Given a σ-algebra $\mathcal{N}$ contained in $\mathcal{M}$, prove that there exists a nonnegative $\mathcal{N}$-measurable function g such that $\int_A g \, d\mu = \int_A f \, d\mu$ for all $A \in \mathcal{N}$. Show that g is unique, up to modification on a μ-null set in $\mathcal{N}$, provided that f is μ-integrable.

Hint. Let μ_f be the measure with μ-density f and apply the Radon-Nikodym theorem to the restrictions of μ_f and μ to $\mathcal{N}$.

Note. If $\mu(X) = 1$, that is, if μ is a probability measure, then g is called the *conditional expectation* of f given $\mathcal{N}$. In the special case $\mathcal{N} = \{\emptyset, X\}$, the function g is a constant (why?), called the *expected value* of f.

6. *Bonus Problem.* Let $n \in \mathbb{N}$. For $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B_n(x, r)$ denote the open ball (with respect to the Euclidean norm) of radius r centered at x.

(a) Let γ_n denote the Lebesgue measure of the unit ball, $B_n(0, 1)$. Show that, for every $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, the Lebesgue measure of $B_n(x, r)$ is $r^n \gamma_n$.

Hint. Consider a suitable mapping $T \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $B_n(x, r) = T(B_n(0, 1))$.

(b) Suppose that f is a locally Lebesgue-integrable function on $\mathbb{R}^n$ such that $\int_{B_n(x,r)} f(y) \, dy = 0$ for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$. Show that $f = 0$ a.e. with respect to Lebesgue measure.