Preliminary Exam in Harmonic Analysis

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1. Give an example of a tight frame that is not an orthonormal basis.

2. If $(f_j)_{j\in J}$ is a tight frame with bound $A = 1$ and, moreover, $\|f_j\| = 1$ for all $j \in J$, prove that $(f_j)_{j\in J}$ is an orthonormal basis.

3. Prove that if S is a positive bounded linear operator on a Hilbert space $\mathcal{H}$, and is bounded below by a strictly positive constant α, then S is invertible and its inverse S^{-1} is bounded by α^{-1}.

4. Let $(f_j)_{j\in J}$ be a frame on a Hilbert space $\mathcal{H}$ with upper frame bound B. Prove that for every $f \in \mathcal{H}$, the frame operator F satisfies $\|Ff\|^2 \leq B\|f\|^2$.

5. State the four conditions that define a multiresolution analysis, and prove that these conditions imply that
$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

6. Let $f \in L^2(\mathbb{R})$. If for $g \in L^2(\mathbb{R})$, $\hat{g}$ is defined by
$$\hat{g}(x) = \int_{\mathbb{R}} g(t)e^{-itx}dt,$$
Prove that $\{f(t-k)\}_{k\in\mathbb{Z}}$ is an orthonormal sequence if and only if
$$\sum_{k\in\mathbb{Z}} |\hat{f}(x+2k\pi)|^2 = 1 \quad a.e.$$

7. Show that in the definition of multiresolution analysis, the condition "$\{f(t-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0" may be replaced by the condition "$\{f(t-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of V_0".

8. Prove that if f is a 2π-periodic, continuous and piecewise smooth function, then the Fourier series of f converges to f absolutely and uniformly in $\mathbb{R}$.

9. Let $B = \{F : \mathbb{D} \to \mathbb{C},\ \textit{analytic in } \mathbb{D} \textit{ such that } \|F\|_B = \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| d\theta < \infty\}$
and $H' = \{F : \mathbb{D} \to \mathbb{C},\ \textit{analytic in } \mathbb{D} \textit{ such that } \|F\|_{H'} = \sup_{0<r<1} \int_0^{2\pi} |F'(re^{i\theta})| d\theta < \infty\}$
Show that B is continuously contained in H', i.e. $B \subseteq H'$ and $\|F\|_{H'} \leq \|F\|_B$.

10. Let f be a bounded function in $[-\pi, \pi]$ and periodic of period 2π. Show that the Fourier coefficient of f, say, a_n and b_n satisfy
$$|a_n| \leq \frac{M}{n} \quad \textit{and} \quad |b_n| \leq \frac{M}{n}.$$

11. Let $L(p, \infty) = \{f : [0, 2\pi] \to \mathbb{R};\ \ \textit{measurable so that}$
$\|f\|_{L(p,\infty)} = \sup_{t>0} t^p \mu\{x \in [0, 2\pi],\ \ |f(x)| > t\} < \infty,\ \ p \geq 1\}$ and
$L^p = \{f : [0, 2\pi] \to \mathbb{R};\ \ \textit{measurable so that}$
$\|f\|_{L^p} = \Big(\int_0^{2\pi} |f(t)|^p \Big)^{1/p} < \infty,\ \ p \geq 1\}.$
Show that $L^p \subseteq L(p, \infty)$ with $\|f\|_{L(p,\infty)} \leq C\|f\|_{L^p}$. Give an example of a function $f \in L(1, \infty)$ so that $f \notin L^1$.

12. Prove that for each m, $\Big| \sum_{n=1}^{m} \frac{\sin nx}{n} \Big| \leq \frac{\pi}{2} + 1.$

13. Define $Lip(\alpha) = \{f : [0, 2\pi] \to \mathbb{R},\ \ \textit{periodic such that}\ \ |f(x+h) - f(x)| \leq Mh^\alpha,\ \ 0 < \alpha < 1\}$ and
$\Lambda_\alpha = \{f : [0, 2\pi] \to \mathbb{R},\ \ \textit{periodic such that}\ \ |f(x+h) - f(x)| \leq Mh^\alpha,\ \ 0 < \alpha < 2\}.$
Show that:

   a. If $\alpha > 1$, then $Lip\alpha = \{0\}$
   b. $Lip(\beta) \subset Lip(\alpha)$ for $\alpha < \beta$.
   c. $Lip(\alpha) \subseteq \Lambda_\alpha$ for $0 < \alpha < 1$.
   d. Can you argue that Λ_α is equivalent as a Banach space to $Lip(\alpha)$ for $0 < \alpha < 1$.