Linear Algebra Prelim, 2003

Professor Peter Nylen

1. Find a 3×3 matrix A with eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_2 = -1$ and corresponding eignevectors $x_1 = [1, 0, 1]^T, x_2 = [1, 0, -1]^T, x_3 = [0, 1, 0]^T$. Explain why there is no matrix with the given λ as eigenvalues and $y_1 = [1, 1, 0]^T, y_2 = [0, 1, 1]^T, y_3 = [1, 0, -1]^T$ the corresponding eigenvectors.

2. Find The Jordan Canonical Form of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}.$$

3. Suppose that $A \in \mathbf{M_{10}}(\mathbf{C})$ has eigenvalue 0 only, and that $\text{rank}(A) = 7$, $\text{rank}(A^2) = 4$, $\text{rank}(A^3) = 2$, and $A^4 = 0$. Find the Rational Canonical Form of A.

4. Let A be $n \times n$ real symmetric with eigenvalues $\lambda_i, i = 1, \ldots, n$.

   (a) Let B be the $n - 1 \times n - 1$ matrix obtained from A by removing the 1^{st} row and column. State the interlacing inequalities that the eigenvalues $\mu_j : j = 1, \ldots, n-1$ of B must satisfy.

   (b) State the majorization inequalities that the diagonal entries of A must satisfy.

5. Find 3×3 symmetric matrix P such that for all $v \in R^3$, Pv is the point on the plane $x + y + z = 0$ closest to v.

6. Let $A, B \in M_n$ be given. Show that if $x^*Ax = x^*Bx$ for all $x \in \mathbf{C}^n$, then $A = B$.

7. State the following:
   (a) Perron's Theorem for positive matrices
   (b) definition of irreducible matrix, and the generalization of Perron's theorem to such matrices.
   (c) The definition of Primitive matrix, and the generalization of Perron's theorem to such matrices.
8. Let A be an irreducible nonnegative matrix. State:
   (a) the defintion of h, the "index of irreducibility".
   (b) conditions equivalent to "A has index of irreducibilty h".
9. Prove: If $A \geq 0$ is $n \times n$ doubly stochastic and irreducible with index of irreducibility h, then n is divisible by h.
10. State the definition of the following terms and phrases:
    (a) norm on R^n
    (b) norm $M_n(R)$ induced by a given norm on R^n
    (c) all norms on R^n are equivalent.
11. Prove that the norm $||\cdot||_2$ on $M_n(R)$ induced by the ℓ_2 (Euclidean) norm is given by the formula $||A|| = \sqrt{\lambda \max(A)}$.
12. Let $||\cdot||$ be a given norm on R^n. Prove that the set $\{x \in R^n : ||x|| \leq 1\}$ is convex.