Linear Algebra Preliminary Exam Summer 2007
Professor Thomas H. Pate

Name:

For full credit show all steps in detail.

1. Give an example of a non-diagonalizable matrix. Prove that your example is non-diagonalizable.

2. Suppose M is an $n \times n$ complex matrix with n distinct eigenvalues. Prove that M is diagonalizable.

3. What do you know about the eigenvalues of the following kinds of matrices?
(a.) unitary, (b.) Hermitian, (c) skew-Hermitian ($A^* = A$).

4. Suppose A is $m \times n$ and B is $n \times m$. What is the relationship between the characateristic polynomial of AB and the characteristic polynomial of BA? For extra credit prove that this relationship exists.

5. Suppose A is an $n \times n$ complex matrix. Prove that there exists a unitary matrix U such that $U^* AU$ is upper triangular.

6. Suppose $A \in \mathbb{C}^{n \times n}$ and let $p(z)$ be the characteristic polynomial of A. Prove that $p(A) = 0$.

7. Suppose $A \in \mathbb{C}^{n \times n}$ and each eigenvalue, λ, of A lies inside the unit circle. Prove that there exists a matrix norm $\| \cdot \|$ such that $\|A\| < 1$.

8. Suppose $A \in \mathbb{C}^{n \times n}$. Prove that $\lim_{k \to \infty} A^k$ exists and is the zero matrix if and only if each eigenvalue of A lies inside the unit circle in the complex plane.

9. Suppose V is a complex vector space and T is a linear map from V to V that is one-to-one. Prove that if $v_1, v_2, \ldots, v_k$ are linearly independent members of V, then $Tv_1, Tv_2, \ldots, Tv_k$ are also linearly independent members of V.

10. Suppose V and W are vector spaces over field $\mathbb{F}$ and V is finite dimensional. Suppose T is a linear map from V to W. Prove that $\dim V = \dim(\text{Ker } T) + \dim(\text{Range } (T))$.

11. Suppose A and B are diagonalizable $n \times n$ complex matrices. When are A and B simultaneously diagonalizable? State and prove a theorem justifying your answer.

12. Suppose T is a linear map from V to V where V is a vector space and let W be a proper invariant subspace of V Assume V is finite dimensional. Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W and extend to get a basis $B = \{w_1, w_2, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ for V. Carefully describe the matrix $M_B^B(T)$, that represents T with respect to B. If $W' = \text{span } \{w_{k+1}, \ldots, w_n\}$ and W' is also invariant under T. Then, what is the appearance of $M_B^B(T)$?

13. Suppose V is a finite dimensional vector space and $T : V \to V$ is linear. Let W be a subspace of V invariant under T. What property must W have in order that there exists a complementary invariant subspace?

14. Suppose $T : V \to V$ is linear and V is an n-dimensional complex vector space. If λ is an eigenvalue of T, then the generalized eigenspace associated with λ is $\mathrm{Ker}\big((T - \lambda I)^n\big)$.

    (a) Prove that generalized eigenspaces of T are invariant under T.

    (b) Prove that if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of T, then the sum $\mathrm{Ker}\big((T-\lambda_1 I)^n\big) + \mathrm{Ker}\big((T-\lambda_2 I)^n\big) + \cdots + \mathrm{Ker}\big((T-\lambda_k I)^n\big)$ is direct.

    (c) Prove that V is the direct sum of the generalized eigenspaces of T.