Linear Algebra Preliminary Exam, 2008
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Name:

For full credit, show all steps in details

Choose 6 out of 7

1. (a) Prove Schur's triangularization theorem by induction: For $A \in M_n(\mathbb{C})$, there is a unitary matrix $U \in M_n$ such that U^*AU is upper triangular.

   (b) Can we get upper triangular form for $A \in M_n(\mathbb{R})$ via real orthogonal matrices similarity? If not, what is the best form?

   (c) Use Schur's triangularization to prove the spectral theorem for Hermitian matrices, i.e., for each Hermitian $A \in M_n$, there is a unitary matrix $U \in M_n$ such that U^*AU is real diagonal.

   (d) Is the spectral theorem for real symmetric matrices also true? i.e., for each $A \in M_n(\mathbb{R})$, there is a real orthogonal matrix O such that O^TAO is real diagonal. Explain.

2. (a) State the theorem of Jordan canonical form on $A \in M_n$.

(b) What are the possible Jordan forms of a matrix $A \in M_n$ such that $A^3 = I$?

(c) If $A \in M_n$ has characteristic polynomial $p_A(t) = (t-3)^3(t-2)^2$ and minimal polynomial $q_A(t) = (t-3)^2(t-2)$, what is the Jordan canonical form of A?

(d) Use the Jordan canonical form to show that $\lim_{m\to\infty} A^m = 0$ if and only if the spectral radius $\rho(A) < 1$. Give two simple examples to show that if $\rho(A) = 1$, A may or may not converge.

3. Let S, T be subspaces of a vector space V.

(a) Use the dimension theorem $\dim(S \cap T) = \dim S + \dim T - \dim(S+T)$ to prove the interlacing inequalities: Let $A \in M_n$ be Hermitian with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and let $B \in M_m$ be a principal submatrix of A with eigenvalues $\beta_1 \geq \cdots \geq \beta_m$. Then

$$\lambda_k \geq \beta_k \geq \lambda_{k+n-m}, \qquad k = 1, \ldots, m.$$

When $m = n-1$, what is the form of the interlacing inequalities?

(b) Define majorization $x \prec y$ where $x, y \in \mathbb{R}^n$.

(c) Use interlacing inequalities and induction to show Schur's theorem: Let $A \in M_n$ be Hermitian. The diagonal $d := (a_{11}, \ldots, a_{nn})^T$ of A is majorized by the eigenvalues $\lambda \in \mathbb{R}^n$ of A.

(d) Is the converse of Schur's theorem true? If so, state it explicitly.

4. (a) State and prove Gersgorin theorem on $A \in M_n$.

   (b) Show that if A is strictly diagonally dominant, then $A \in M_n$ is nonsingular.

   (c) Prove that if $A \in M_n$, then $\rho(A) \le \min\{\max_i \sum_{j=1}^n |a_{ij}|, \max_j \sum_{i=1}^n |a_{ij}|\}$.

   (d) Use (c) to show that $|\det A| \le \min\{\prod_{i=1}^n (\sum_{j=1}^n |a_{ij}|), \prod_{j=1}^n (\sum_{i=1}^n |a_{ij}|)\}$.

5. (a) Define the notion of dual norm of a norm $\| \cdot \|$ on $\mathbb{C}^n$. Show that vector norms $\| \cdot \|_1$ and $\| \cdot \|_\infty$ on $\mathbb{C}^n$ are dual to each other.

(b) What is the difference between a matrix norm $|\!|\!| \cdot |\!|\!|$ on M_n and a vector norm on M_n? Give an example that is a vector norm on M_n but not a matrix norm.

(c) Show that if $|\!|\!| \cdot |\!|\!|$ is a matrix norm on M_n, then $\rho(A) \leq |\!|\!| A |\!|\!|$.

(d) Prove Gelfand's spectral theorem: $\rho(A) = \lim_{k \to \infty} |\!|\!| A^k |\!|\!|^{1/k}$ for any matrix norm $|\!|\!| \cdot |\!|\!|$ (you may use Question 2(d))

6. (a) Let $A \in M_{m,n}$ with $m \le n$. Prove that SVD ($A = V\Sigma W^*$, $V \in M_m$, $W \in M_n$ unitary, Σ "diagonal") and polar decomposition ($A = PU$, $P \in M_m$ positive semidefinite and $U \in M_{m,n}$ has orthonormal rows) are equivalent.

(b) Prove either the above SVD or polar decomposition.

(c) Show that if $\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$ where $S = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ and r is the rank of $A \in M_{m,n}$, then $A = V_1 S W_1^*$ where $V = [V_1 \; V_2]$, $V_1 \in M_{m,r}$ and $W = [W_1 \; W_2]$, $W_1 \in M_{n,r}$.

(d) Compute SVD of a nonzero vector $x \in M_{n,1}(\mathbb{C})$.

7. (a) Define irreducible $A \in M_n$. Then give three equivalent conditions of irreducibility.

   (b) State Perron theorem (6 statements) on positive matrices $A \in \mathbb{R}_{n\times n}$. When $A \in \mathbb{R}_{n\times n}$ is irreducible nonnegative, which statements are true (i.e., Frobenius theorem)? Give counterexamples to those not true.

   (c) Suppose that $A \in M_n(\mathbb{R})$ is irreducible nonnegative and that $B \geq 0$ commute with A. If x is the Perron vector of A, prove that $Bx = \rho(B)x$.

   (d) What can we say about the 7 eigenvalues of an irreducible nonnegative and nonsingular $A \in M_7(\mathbb{R})$? Why?