Dr. Zenor's study help for MH 650-2

Definition 1 *Let X be a point set. The ordered pair $(X, \mathcal{T})$ is called a topological space if*

1. *$\mathcal{T}$ is a collection of subsets of X.*
2. *X is in $\mathcal{T}$ and the empty set is in $\mathcal{T}$.*
3. *$\mathcal{T}$ is closed under unions and finite intersections.*

The members of $\mathcal{T}$ are called open sets.

Definition 2 *P is a* limit point *of the set H if every open set containing P contains a point of H distinct from P.*
A set H is closed *provided it contains all of its limit points.*
The interior of a set H is the union of all open subsets of H. The set N is a neighborhood of the point x if x is in the interior of N.

Question 1 *Must a finite set be closed?*

Theorem 1 *The union of two closed sets is closed.*

Theorem 2 *If $\mathcal{H}$ is a collection of closed sets with a common part, then that common part is closed.*

Definition 3 *A point P is a* boundary point *of the set H provided that every open set containing P contains a point in H and a point in $X - H$.*

Theorem 3 *If R is an open set and p is in R, then p is not a boundary point of R.*

Question 2 *Must the common part of a collection of open sets be open?*

Question 3 *If a set is not open, must it be closed.*

Definition 4 *If H is a set, then the closure of H is H together with all of its limit points. We will denote the closure of H by $cl(H)$.*

Theorem 4 *If H is a set, then $cl(H)$ is closed.*

Definition 5 *The space X is T_1 if p and q are distinct points, then there is an open containing p but not q.*

Theorem 5 *Finite subsets of a T_1 space are closed.*

Definition 5.5: Suppose that $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ is a subset of $\mathcal{T}$ such that if U is an open set containing the point p, then there is a member B of $\mathcal{B}$ such that $p \in B \subset U$. Then $\mathcal{B}$ is called a *basis* for $\mathcal{T}$.

Question 4 *Is $cl(H) \cup cl(K) = cl(H \cup K)$?*

Question: Is $cl(H) \cap cl(K) = cl(H \cap K)$?

Definition 6 *The space X is* Hausdorff *if it is true that if p and q are distinct points, then there are mutually exclusive open sets U and V such that $p \in U$ and $q \in V$.*

Question 5 *Is there a T_1–space that is not Hausdorff?*

Unless otherwise stated, we will assume that our topological spaces are Hausdorff.

Definition 7 *The collection of sets $\mathcal{U}$* covers *the set H if each point of H is contained in a member of $\mathcal{U}$.*

Definition 8 *The set H is* compact *if it is true that whenever $\mathcal{U}$ is a collection of open sets covering H, then some finite subcollection of $\mathcal{U}$ covers H.*

Theorem 6 *If H is a compact subset of X, then H is closed.*

Theorem 7 *If H is a compact subset of X, then every infinite subset of H has a limit point.*

Definition 9 *The collection of sets H is* centered *if every finite subcollection of H has a point in common.*

Theorem 8 *The set H is compact if and only if every centered collection of closed subsets of H has a nonempty intersection.*

Definition 10 *The collection of sets $\mathcal{H}$ is monotone if whenever A and B are in $\mathcal{H}$, then one is a subset of the other.*

Theorem 9 *The space X is compact if and only if every monotone collection of nonempty closed subsets of X has a nonempty intersection.*

Question 6 *If every infinite subset of X has a limit point, must X be compact?*

AXIOM Every set can be well ordered.

Question 7 *Is ω_1 compact?*

Question 8 *Is $\omega_1 + 1$ compact?*

Definition 11 *A* metric function *on a set X is a function $\rho : X \times X \to [0, \infty)$ such that*

1. *$\rho(x, y) = 0$ if and only if $x = y$.*
2. *$\rho(x, y) = \rho(y, x)$*
3. *$\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.*

A metric space is an ordered pair (X, ρ), where ρ is a metric function on X. We will speak of a metric space X with the existence of the metric function ρ understood.

If ρ is a metric function on X and $x \in X$, then we will denote by $B_\epsilon(x)$ the set $\{y \in X | \rho(x, y) < \epsilon\}$. We will say that the topological space X is metrizable provided that there is metric ρ on X, such that $\{B_\epsilon(x) | x \in X \text{ and } \epsilon > 0\}$ is a basis for the topology on X.

If p is a point in the metric space X and H is a subset of X, then p is a limit point of H if for every $\epsilon > 0$, there is a point q of H such that $0 < \rho(p, q) < \epsilon$.

Question 9 *How many functions are there from ω into ω?*

Definition 12 *A space X is* regular *(T_3) if whenever U is an open set containing the point p, there is an open set V containing p such that $cl(V) \subset U$.*

Definition 13 *The space X is locally compact provided that for every p in X, there is an open set U containing p such that $cl(U)$ is compact*

Theorem 10 *A locally compact Hausdorff space is regular.*

Theorem 11 *If H and K are mutually exclusive compact subsets of X, then there are mutually exclusive open sets, one containing H and the other containing K.*

Theorem 12 *The plane is locally compact.*

Question 10 *Is ω_1 locally compact?*

Definition 14 *A space X is Lindelöf if every open cover of X has a countable subcover.*

Definition 15 *The space X is normal if whenever H and K are mutually exclusive closed sets, there are mutually exclusive open sets U and V such that $H \subset U$ and $K \subset V$.*

Theorem 13 *Every regular Lindelöf space is normal*

Theorem 14 *Every metrizable space is normal.*

Definition 16 *If X and Y are topological spaces, then the function $f : X \to Y$ is continuous provided that if U is an open set in Y containing $f(p)$, then there is an open set V in X containing p such that $f(V) \subset U$.*

Theorem 15 *The space is normal if and only if it is true that if H and K are mutually exclusive closed sets, then there is a continuous function $f : X \to [0,1]$ such that $H \subset f^{-1}(0)$ and $K \subset f^{-1}(1)$.*

Theorem 16 *If X is normal, H is a closed subset of X, and $f : H \to [0,1]$ is continuous, then there is a continuous $F : X \to [0,1]$ such that $f(x) = F(x)$ for every x in H.*

Definition 17 *The set $H \subset X$ is* dense *in X if $X = cl(H)$. The space X is* separable *if there is a countable dense subset of X.*

Theorem 17 *If X is metrizable, then the following are equivalent:*

1. *X is Lindelöf.*
2. *X has a countable basis.*
3. *Every uncountable subset of X has a limit point.*
4. *X is separable.*
5. *Every uncountable subset of X has a limit point in itself.*

Theorem 18 *If X is metrizable, then the subset K of X is compact if and only if K is closed and every infinite subset of K has a limit point.*

Definition 18 *The subset H of X is a* zero *set if there is a continuous function $f : X \to [0,\infty)$ such that $H = f^{-1}(0)$.*

Theorem 19 *If H is a zero set in X, then H is closed.*

Theorem 20 *If X is metrizable then every closed subset of X is a zero set.*

Question 11 *Must the common part of two zero sets be a zero set?*

Question 12 *Must the union of two zero sets be a zero set?*

Question 13 *Is every subset of a Lindelöf space Lindelöf?*

The Tangent circle space. $X = \{(x,y) \in \mathcal{R}^2 | y \geq 0\}$ If $y > 0$, then a basic open set containing (x,y) is the interior of a disk containing (x,y). If $y = 0$, and if $\epsilon > 0$, then $\{(u,v) : \rho((u,v),(x,\epsilon)) < \epsilon\} \cup \{(x,0)\}$ is a basic open set containing $(x,0)$.
Sorgenfrey's line: The real line where basic open sets are sets of the form $[a,b)$.

Question 14 *Which of the following properties does the Tangent Circle space have:*

1. *Separable?*
2. *Lindelöf?*
3. *Normal?*
4. *Regular?*
5. *Completely regular?*
6. *Locally compact?*
7. *Developable?*
8. *Strongly developable?*
9. *Metrizable?*
10. *Paracompact?*
11. *Countable basis?*

Question 15 *Which of the following properties does the Sorgenfrey line have:*

1. *Separable?*
2. *Lindelöf?*
3. *Normal?*
4. *Regular?*
5. *Completely regular?*
6. *Locally compact?*
7. *Developable?*
8. *Strongly developable?*
9. *Metrizable?*
10. *Paracompact?*

11. *Countable basis?*

Question 16 *If A is a well-ordered set and $A' = a_1 > a_2 > \dots$ is a subset of A, then how large (in cardinality) can A be? (i.e., can it be ifninite?).*

Definition 19 *A set $H \subset X$ is a $G_\delta-$set provided that there is a sequence of open sets $U_1, U_2, \dots$ such that $H = \cap_{i=1}^{\infty} U_i$. $H \subset X$ is a regular $G_\delta-$set provided that there is a sequence of open sets $U_1, U_2, \dots$ such that $H = \cap_{i=1}^{\infty} U_i = \cap_{i=1}^{\infty} \bar{U}_i$.*

Definition 20 *The space X is perfectly nomal if X is normal and each closed subset of X is a $G_\delta-$set.*

Theorem 21 *The continuous image of a compact space is compact.*

Theorem 22 *The following statements are equivalent:*

1. *X is perfectly normal.*
2. *every closed subset of X is a regular $G_\delta-$set.*
3. *Every closed subset of X is a zero set.*

Definition 21 *If $\mathcal{U}$ and $\mathcal{V}$ are collections of sets, then $\mathcal{U}$ refines $\mathcal{V}$ if for every $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $U \subset V$.*

Definition 22 *The collection of sets $\mathcal{U}$ is locally finite if, for every x, there is an open set W containing x such that $\{U \in \mathcal{U} | W \cap \mathcal{U}\}$ is finite.*

Definition 23 *The space X is paracompact if for each open cover $\mathcal{U}$ of X, there is a locally finite open refinement of $\mathcal{U}$ covering X.*

Theorem 23 *Every metric space is paracompact.*

Theorem 24 *Every paracompact space is normal.*

Theorem 25 *Every regular Lindelöf space is paracompact.*

Theorem 26 *The space X is compact if and only if every infinite subset of X has a limit point and X is paracompact.*

Theorem 27 *The regular space X is Lindelöf if and only if every uncountable subset of X has a limit point and X is paracompact.*

Question 17 *Which of the following properties does ω_1 have:*

1. *Separable?*
2. *Lindel"̈of?*
3. *Paracompact?*
4. *Normal?*
5. *Perfectly normal?*
6. *Locally compact?*

7. *Developable?*

8. *Strongly developable?*

Theorem 28 *Suppose that $f : \omega_1 \to \omega_1$ is such that $f(\alpha) < \alpha$ for all $\alpha \in \omega_1$. Then there is a γ such that $f(\alpha) = \gamma$ for uncountably many α.*

Question 18 *Is every subspace of a paracompact space paracompact?*

Definition 24 *The sets H and K are* mutually separated *if $\bar{H} \cap K = H \cap \bar{K} = \emptyset$.*

The set H is connected if it is not the sum of two non-empty mutually separated sets. A compact and connected set is called a continuum.

If H and K are sets, then the continuum C is irreducible *from H to K if C intersects both H and K and no proper subcontinuum intersects both H and K.*

If H is a set and $x \in H$, then the component of x in H is the union of all the connected subsets of H that contains x.

Theorem 29 *If $\mathcal{H}$ is a collection of connected sets with a point in common, then $\cup\mathcal{H}$ is connected.*

Theorem 30 *The continuous image of a connected set is connected.*

Theorem 31 *If $\mathcal{H}$ is a monotonic collection of continuua, then $\cap\mathcal{H}$ is connected.*

Theorem 32 *If C is irreducible from the closed sets H and K, then every point of $H \cap C$ is a limit point of $C - H$.*

Theorem 33 *If C is connected, then $\bar{C}$ is connected.*

Theorem 34 *Suppose that H and K are mutually exclusive closed subsets of the compact set M and M cannot be divided into mutually exclusive closed sets A and B, one containing H and the other containing K. Then M contains a continuum C which is irreducible from H to K.*

Theorem 35 *Suppose that C is a continuum, U is an open set containing $x \in C$, where $C - U \neq \emptyset$, and K is the component of x in $C \cap U$. The the boundary of U contains a limit point of K.*

Definition 25 *If X and Y are topological spaces, then $\mathcal{B} = \{U \times V | U$ is open in X and V is open in $Y\}$ is a basis for the* product topology *on $X \times Y$.*

Question 19 *If X and Y have property $\mathcal{P}$, then must $X \times Y$ have property $\mathcal{P}$?*

a. *$\mathcal{P}$: Compact*
b. *$\mathcal{P}$: connected?*
c. *$\mathcal{P}$: normal?*
d. *$\mathcal{P}$: Lindelöf?*
e. *$\mathcal{P}$: paracompact?*
f. *$\mathcal{P}$: metrizable?*
g. *$\mathcal{P}$: countable bases?*
h. *$\mathcal{P}$:separable?*
i. *$\mathcal{P}$: regular?*
j. *$\mathcal{P}$: Hausdorff?*
k. *$\mathcal{P}$: Completely regular?*
l. *$\mathcal{P}$: locally compact?*
m. *$\mathcal{P}$: closed sets are $G_\delta-$sets.?*
n. *$\mathcal{P}$: developable?*
o. *$\mathcal{P}$: strongly developable.*

Definition 26 *A set H is perfect if every point of H is a limit point of H.*

Theorem 36 *Every compact, perfect space is uncountable.*

Theorem 37 *Suppose that C is a compact, perfect, metric space with a basis of open and closed sets. Then C is homeomorphic to the Cantor set.*

Question 20 *Is the product of the Cantor set with itself homeomorphic to the Cantor set?*

Question 21 *Is every ordered space normal?*

Definition 27 *Suppose that (X, ρ) is a metric space and that $\{x_n\}$ is a sequence of points in X. Then $\{x_n\}$ is* Cauchy *if, for every $\epsilon > 0$, there is an integer N such that if $n > N$ and $m > N$, then $\rho(x_n, x_m) < \epsilon$. The space X is completely metrizable if there is a metric ρ on X, compatible with the topology on X, such that every Cauchy sequence converges.*

Theorem 38 *Every compact metrizable space is completely metrizable.*

Definition 28 *The set $H \subset X$ is nowhere dense provided that $\bar{H}$ contains no open set.*

Theorem 39 *If $H \subset X$ is nowhere dense, then $X - \bar{H}$ is a dense open set.*

Theorem 40 *If X is completely metrizable, then X is not the sum of countably many nowhere dense sets.*

Theorem 41 *If X is locally compact, then X is not the sum of countably many nowhere dense sets.*

Theorem 42 *If X a $G_\delta-$set in a compact space, then X is not the sum of countably many nowhere dense sets.*

Definition 29 *The space X is completely regular provided that if U is an open set containing $x \in X$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $(X - U) \subset f^{-1}(1)$.*

Definition 30 *The space X is* developable *if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of collections of open sets covering X such that*

1. *$\mathcal{G}_{n+1} \subset \mathcal{G}_n$ for each n.*
2. *If U is an open set containing x, then there is an n such that if $x \in G \in \mathcal{G}_n$, then $G \subset U$.*

The sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ is called a development *for the topology on X.*

The space X is strongly developable *if there is a development $\mathcal{G}_1, \mathcal{G}_2, \ldots$ for X such that if U is an open set containing x, then there is an n such that if G_1 and G_2 are in $\mathcal{G}_n$, $x \in G_1$ and $G_1 \cap G_2 \neq \emptyset$, then $G_1 \cup G_2 \subset U$.*

Theorem 43 *If H is a subset of the developable space X, then H is compact if and only if H is closed and every infinite subset of H has a limit point.*

Theorem 44 *The developable space X is Lindelöf if and only if every uncountable subset of X has a limit point.*

Theorem 45 *If X is strongly developable, then X is normal.*

Exercise 1 *Let $\mathcal{U} = \{U_\alpha | \alpha < \gamma\}$ be a well ordered open cover of the developable space X. Let $\{\mathcal{G}_n\}$ be a development for X. For each n and for each $\alpha < \gamma$, let $H_{(\alpha,n)} = \{x \in U_\alpha | st(x, \mathcal{G}_n) \subset U_\alpha\}$ and let $K_{(\alpha,n)} = H_{(\alpha,n)} - \cup_{\beta<\alpha} U_\beta$. Show that*

1. *Show that $\mathcal{K} = \{K_{(\alpha,n)} | n < \omega, \alpha < \beta\}$ covers X.*
2. *Show that $\mathcal{K} = \{K_{(\alpha,n)} | n < \omega, \alpha < \beta\}$ refines $\mathcal{U}$.*

3. *Show that for each n, if $A \subset \beta$, then $\cup\{K_{(\alpha,n)}|\alpha \in A\}$ is closed.*

4. *Show that for each n, if $\alpha_1 \neq \alpha_2$, then $K_{(\alpha_1,n)} \cap K_{(\alpha_2,n)} = \emptyset$.*

Exercise 2 *Suppose that H and K are mutually exclusive closed sets and $\{U_n\}_{n<\omega}$ and $\{V_n\}_{n<\omega}$ are open covers of H and K, respectively, such that, for each n, $\bar{U_n} \cap K = \emptyset$ and $\bar{V_n} \cap H = \emptyset$. Then there are mutually exclusive open sets U and V, with $H \subset U$ and $K \subset V$.*

Definition 31 *The space X is* collectionwise normal *if it is true that if $\mathcal{H}$ is a discrete collection of closed sets, then there is a collection of mutually exclusive open sets $\{O(H)|H \in \mathcal{H}\}$ such that $H \subset O(H)$.*

Theorem 46 *A paracompact space is collectionwise normal.*

Theorem 47 *A strongly developable space is collectionwise normal.*

Theorem 48 *If X is collectionwise normal and $\mathcal{H}$ is a discrete collection of closed sets, then there is a discrete collection of open sets $\{O(H)|H \in \mathcal{H}\}$ such that $H \subset O(H)$, for each $H \in \mathcal{H}$.*

Theorem 49 *A strongly developable space is paracompact.*

Theorem 50 *A strongly developable space is metrizable.*

Definition 32 *$H \subset X$ is a* retract *of X if there is a continuous function $r : X \to H$ such that $r(x) = x$ for all $x \in H$.*

Theorem 51 *A retract of X is closed.*

Lemma: If X is normal and I is a homeomorphic image of [0,1] lying in X, then I is a retract of X.

Theorem 52 *If K is a homeomorphic copy of $[0,1]^n$ lying in the normal space X, then K is a retract of X.*

Definiton A 1 *Given that* $f, g : [0,1] \to X$ *are continuous functions such that* $f(0) = f(1) = p_0 = g(0) = g(1)$ *then* $f \sim g$ *means that there is a continuous function* $F : [0,1] \times [0,1] \to X$ *such that* $F(x,0) = f(x)$ *and* $F(x,1) = g(x)$ *and* $F(0,x) = F(1,x) = p_0$*.*

Theorem A 1 $f \sim g$ *is an equivalence relation.*

Definiton A 2 $[f \oplus g] = [f] \oplus [g]$ *where* $f \oplus g = \begin{cases} f(2x) & 0 \leq x \leq 1/2 \\ g(2x-1) & 1/2 \leq x \leq 1 \end{cases}$

Theorem A 2 $[f] \oplus [id] = [f]$*.*

Question A 1 $[f \oplus g] = [g \oplus f]$*?*

Question A 2 *What does* $[-f]$ *equal?*

Theorem A 3 *Show that* $[f] \oplus [g] = [f \oplus g]$ *is well-defined.*

Theorem A 4 *If* $f \sim -f$ *then* $f \sim id$*.*

Theorem A 5 *The operation* $\oplus$ *is associative.*

Theorem A 6 $[f] \oplus [-f] = [id]$ *and* $[-f] \oplus [f] = [id]$*.*

Theorem A 7 *If* H *is a retract of* Y *and* $x_0 \in H$ *then* $H_1(X, x_0)$ *is epimorphic to* $H_1(H, x_0)$*.*

Theorem A 8 $H_1(X, x_0) \oplus H_1(Y, y_0)$ *is isomorphic to* $H_1(X \times Y, (x_0, y_0))$*.*