Math 7800-7810 Probability Theory Prelim Exam
August 19, 2011

Instructor: Erkan Nane
Time: 9:00am–1:00pm

Your goal in this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete. The exam has twelve questions. Choose 8 of them as outlined below.

Do two of the problems 1, 2 or 3

Problem 1. Let X and Y be independent random variables and let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be measurable functions. Prove that if $E|f(X) - g(Y)| < \infty$ then $E|f(X)| < \infty$ and $E|g(Y)| < \infty$.

Problem 2. State and prove the First and Second Borel-Cantelli Lemmas.

Problem 3. Let $\{A_n\}$ be a sequence of independent events. Assume that

$$\sum_{n=1}^{\infty} P(A_n) = \infty.$$

Let A be an event such that

$$\sum_{n=1}^{\infty} P(A_n \cap A) < \infty.$$

Show that $P(A) = 0$.

Do two of the problems 4, 5 or 6

Problem 4. If $X_1, X_2, \cdots$ are integrable, and possibly dependent but have the same distribution, show that as $n \to \infty$,

$$\frac{1}{n} \max_{1 \le k \le n} X_k \to 0, \text{ in probability}$$

and

$$\frac{1}{n} E(\max_{1 \le k \le n} X_k) \to 0.$$

Problem 5. Prove that if X is measurable $\mathcal{F}$, and if X and XY are integrable, then

$$E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$$

Problem 6. (a) State without proof the martingale convergence theorem.
(b) Let $\{\xi_j : j = 0, 1, \cdots\}$ be i.i.d., mean zero, variance one, with all moments finite. If $\sum_{n=1}^{\infty} a_n^2 < \infty$, prove that

$$\sum_{n=1}^{\infty} a_n \xi_1 \xi_2 \cdots \xi_n = a_1\xi_1 + a_2\xi_1\xi_2 + a_3\xi_1\xi_2\xi_3 + \cdots$$

converges almost surely.

Do two of the problems 7, 8 or 9

Problem 7. Suppose that $X_1, X_2, \cdots$ is a sequence of IID random variables and assume that $E(X_1) = 0$ and $E(X_1^2) = 1$.
a. Prove that $\frac{X_n}{\sqrt{n}} \to 0$ as $n \to \infty$, a.s. Then show that in fact

$$\frac{\max_{1\le k\le n}\{|X_k|\}}{\sqrt{n}} \to 0, \quad \text{as } n \to \infty, \ a.s.$$

Hint for the second part: Show that for any sequence of numbers $\{a_n\}_{n\ge1}$:

$$\frac{a_n}{\sqrt{n}} \to 0, \text{as } n \to \infty \text{ implies } \frac{\max_{1\le k\le n}\{|a_k|\}}{\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

b. Let $X_{k,n} = X_k 1_{\{|X_k| < \sqrt{n}/2\}}$, $1 \le k \le n$. Prove that

(i) $\sum_{k=1}^{n} \frac{X_{k,n}}{\sqrt{n}} - \sum_{k=1}^{n} \frac{X_{k,n}^2}{2\cdot n}$ converges in distribution to $Y \sim N(-\frac{1}{2}, 1)$. Hint: You may use the CLT for $\{X_n\}_{n\ge1}$ and part a.

(ii)

$$\sum_{k=1}^{n} \frac{|X_{k,n}|^3}{n^{3/2}} \to 0 \quad \text{as} \quad n \to \infty, a.s.$$

c. Prove that $\prod_{k=1}^{n}(1 + \frac{X_k}{\sqrt{n}})$ converges in distribution to e^Y where $Y \sim N(-\frac{1}{2}, 1)$. Hint: Uset the following inequality (follows from Taylor's expansion)

$$|\log(1+y) - y + \frac{y^2}{2}| \le |y|^3, \ |y| < 1/2$$

Observe that we may have $\frac{X_k}{\sqrt{n}} < -1$.

Problem 8. Let $X_k \sim Uniform[-k, k]$ for $k = 1, 2, \cdots$ and assume that $X_1, X_2, \cdots$ are independent. Does

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} var(X_k)}}$$

converge in probability? Does it converge in distribution? Justify your answers.

Problem 9.(a) Let $X_1, X_2, \cdots$ be iid, $P(X_i = 0) = P(X_i = 1) = \frac{1}{2}$. Let $Y_n = \prod_{i=1}^{n} X_i$. Are the random variables $Y_n, n \ge 1$, uniformly integrable?
(b) Same question, but with $X_i's$ iid $Uniform[0, 2]$

(c) Same question, but with $X_i's$ iid $Uniform[0, 3/2]$

Do two of the problems 10, 11 and 12

Problem 10. State the Kolmogorov extension theorem. Explain how to use Kolmogorov extension theorem to construct Brownian motion.

Problem 11. Let B_t be a standard Brownian motion on the real line. Let $T_0 = \inf\{s > 0 : B_s = 0\}$ and let $R = \inf\{t > 1 : B_t = 0\}$. R is for right or return. Use the Markov property at time 1 to get

$$P_x(R > 1 + t) = \int_{-\infty}^{\infty} p_1(x, y) P_y(T_0 > t) dy$$

where $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$

Problem 12. Let $T_0 = \inf\{s > 0 : B_s = 0\}$ and let $L = \sup\{t \leq 1 : B_t = 0\}$. L is for left or last. Use the Markov property at time $0 < t < 1$ to conclude

$$P_0(L \leq t) = \int_{-\infty}^{\infty} p_t(0, y) P_y(T_0 > 1 - t) dy$$