STAT 7600/7610 Mathematics Statistics Preliminary Exam
August 14, 2015

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Name:

1. It is a closed-book in-class exam.
2. Calculator is allowed.
3. Show your work to receive full credits. Highlight your final answer.
4. Solve any five problems out of eight.
5. Total points are 50 with 10 points for each problem.
6. Time: 180 minutes. (8:00am - 11:00am, August 14th, 2015)

1	2	3	4	5	6	7	8	Total

1. (a) Let $X_1, \ldots, X_n$ be an iid sample from a population with density $f(x)$ and cdf $F(x)$. The order statistics are $X_{(1)}, \ldots, X_{(n)}$. Let $T = X_{(k)} - X_{(k-1)}$ and find the density of T.

   (b) Find the density of $T = X_{(k)} - X_{(k-1)}$ assuming the density is $f(x) = 1$ for $0 < x < 1$.

2. Let $X_1, \ldots, X_n$ be a random sample from the parent population

$$f_X(x;\theta) = \frac{1}{2}e^{-|x-\theta|}, -\infty < \theta < \infty, -\infty < x < \infty.$$

Consider the random variable

$$U_n = \frac{\sum_{i=1}^n X_i - n\theta}{\sqrt{2n}}$$

(a) Derive an explicit expression for the moment generating function of the random variable U_n.

(b) Find the asymptotic distribution of U_n.

3. Suppose that the random variable X represents the time (in months) from the initial diagnosis of leukemia until the first chemotherapy treatment, and the random variable Y represent the time (in months) from the initial diagnosis of leukemia until death. The joint density function of these X and Y random variables is given as

$$f_{X,Y}(x, y) = 2\theta^{-2}e^{-(x+y)/\theta}, \quad 0 < x < y < \infty, \quad \theta > 0.$$

Prove that

$$E(Y^r \mid X = x) = \sum_{j=0}^{r} C_j^r x^{r-j}\Gamma(j + 1)\theta^j$$

for a r non-negative integer.

4. Two continuous random variables have the joint density function

$$f_{X,Y}(x, y; \theta) = e^{-(\theta x + \theta^{-1} y)}, \quad x > 0, y > 0, \theta > 0.$$

Let (X_i, Y_i), $i = 1, \ldots, n$ constitute a random sample of size n from $f_{X,Y}(x, y; \theta)$. Consider estimating θ using

$$\hat{\theta} = (\bar{Y}/\bar{X})^{1/2},$$

where $\bar{Y} = n^{-1} \sum_{i=1}^{n} Y_i$ and $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

(a) Find $E(\hat{\theta})$.

(b) Is $\hat{\theta}$ an unbiased estimator of θ?

5. The independent random variables $X_1, \ldots, X_n$ have the common distribution

$$P(X_i \leq x \mid \alpha, \beta) = \begin{cases} 0 & x < 0 \\ (x/\beta)^\alpha & 0 \leq x \leq \beta \\ 1 & x > \beta, \end{cases}$$

where the parameters α and β are positive.

(a) Find a two dimensional sufficient statistic for (α, β).

(b) Find the maximum likelihood estimates for α and β.

(c) If α is a known constant, α_0, find an upper confidence limit for β with confidence coefficient 0.95.

6. Let $X_1, \ldots, X_n$ be an iid sample from $\mathrm{N}(\mu_x, \sigma_x^2)$ and $Y_1, \ldots, Y_m$ be an iid sample from $\mathrm{N}(\mu_y, \sigma_y^2)$. Assume $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are independent. Assume all parameters $\mu_x, \mu_y \in R; \sigma_x, \sigma_y \in (0, \infty)$ are unknown and $n > 1$ and $m > 1$.

   (a) Find the UMVUE of $\mu_x - \mu_y$.

   (b) If $\sigma_x = \sigma_y = \sigma$, find the UMVUE of $\mu_x - \mu_y$ and σ, and determine if these achieve the CR lower bound.

7. Let $X_1, \ldots, X_n$ be a random sample of size n from a population with unknown mean μ and unknown variance σ^2. Show that there exists a set of values for k, $0 < k < 1$, such that the estimator $k\bar{X}$ has smaller mean-squared error (MSE) as an estimator of μ than does the sample mean $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

8. Let $X_1, \ldots, X_n$ be an iid $\mathrm{N}(\mu, \sigma^2)$, where μ_0 is a specified value of μ and σ^2 is unknown. We would like to test

$$H_0 : \mu = \mu_0 \quad vs. \quad H_1 : \mu \neq \mu_0$$

(a) Find the likelihood ratio test (identify test statistic (T) and rejection region).

(b) Find the sampling distribution of the statistic you find in part (a).

(c) Is it UMP test? Why?