Topology Preliminary Examination March 31, 1994

Hand in no more than six of the following (and at least six if possible). In each problem, carefully define each term in the statement unless you have defined it already in an earlier problem. For instance, in problem 1, "$T_2$ space", "compact", and "closed" should be defined.

Each proof should be from scratch. That is, please do not quote some powerful theorem that you may have learned, in 650-651 or elsewhere, but rather use only the relevant definitions and any technique you know. Exception: feel free to use Zorn's Lemma, or the Axiom of Choice, or the Well-Ordering Axiom.

1. Prove or disprove: If $(X, T)$ is a $T_2$ space, then each compact subset of $X$ is closed.

2. Prove or disprove: If $(X, T)$ is a connected $T_2$ space, $C$ is a connected subset of $X$, and $X - C = H \cup K$, where $H$ and $K$ are disjoint separated sets, then $C \cup H$ is connected.

3. Suppose that $(X, T)$ is a topological space. Then:

   (a) Prove or disprove: If each infinite subset of $X$ has a limit point, then $X$ is compact.

   (b) Prove or disprove: If $X$ is compact, then each infinite subset of $X$ has a limit point.

4. Prove or disprove: Suppose $(X, T)$ is a $T_2$ space and suppose that if $G$ is any open covering of $X$ with the finite intersection property, then there is a point of $X$ common to each element of $G$. Then $X$ is compact.

5. Suppose that $(X, T)$ and $(Y, S)$ are topological spaces and f is a continuous function from $X$ onto $Y$. Amongst the possibilities of $J$ is $T_0$, $J$ is $T_1$, $J$ is $T_2$, $J$ is $T_4$, or $J$ is metric, for each statement below, either prove the statement using the weakest possible $J$, or disprove the statement with an example satisfying the strongest possible $J$.

   (a) If $X$ is compact, then $Y$ is compact.

(b) If $Y$ is compact, then $X$ is compact.

6. Prove or disprove: If $(X, T)$ and $(Y, S)$ are connected $T_2$ spaces, then their product $X \times Y$, with the usual product topology, is also connected.

7. Prove or disprove: If $(X, T)$ is a $T_2$ space, then each component of $X$ is closed.

8. Prove or disprove: Suppose $(X, T)$ is a $T_2$ space, and $A$ and $B$ are subsets of $X$ such that both $A \cup B$ and $A \cap B$ are connected. Then $A$ is connected.