Topology Prelimary Examination, April 21, 2001. Time allotted to this exam: 2 hrs 30 min.

I. Definitions. Define each of the following notions:

1. neighborhood of a point
2. Hausdorff space
3. metric
4. first countable space
5. second countable space
6. compact space
7. connected space

II. Problems: Do at least 7 of the 10 problems; 2 counts as 1 problem and 9 counts as 1 problem. X and Y stand for topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$, respectively.

1. Suppose M and N are subsets of X. Prove that if M contains all of its limit points and N contains all of its limit points, then $M \cup N$ contains all of its limit points.
2. (a) If A is a dense subset of X and U is open in X, then $\overline{(A \cap U)} \supset U$.
   (b) If A is a dense locally compact subspace of a Hausdorff space X, then A is open in X.
3. Suppose that A is a connected subset of X. Show that if $A \subset B \subset \overline{A}$, then B is also connected.
4. Show that if $f : X \to Y$ is a continuous function and $A \subset X$, then $f(\overline{A}) \subset \overline{f(A)}$.
5. Prove that the continuous image of a compact space is compact.
6. Suppose that $X_1 \supset X_2 \supset X_3 \supset \cdots$ is a nested sequence of continua in X. Show that their intersection is connected.
7. If A is a compact subspace of X, B is a compact subspace of Y, and U is an open subset of $X \times Y$ containing $A \times B$, then there exist open subsets V, W of X, Y respectively, such that $A \subset V, B \subset W$, and $V \times W \subset U$.
8. If X is a normal space, and $A_1, A_2, \ldots, A_n$ is a finite sequence of closed subsets of X such that $\bigcap_{j=1}^{n} A_j = \emptyset$, then there exist open subsets $U_1, U_2, \ldots, U_n$, such that $A_j \subset U_j$ for $j = 1, 2, \ldots, n$ and $\bigcap_{j=1}^{n} U_j = \emptyset$.
9. Let X be the interval (0,1) with the usual topology.
   (a) Prove that every open set in X is the union of countably many open intervals.
   (b) Give an example of an open set in X that is not the union of finitely many open intervals.
10. Prove that the interval [0,1] (with the usual topology) and the set

$$W = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1, y = \sin \frac{1}{x}\} \cup \{(x, y) \in \mathbb{R}^2 \mid x = 0, -1 \leq y \leq 1\}$$

are not homeomorphic.