Topology Prelim Exam, August 29, 2011

Do problems 1 and 2 and then five of the remaining problems 3 -9. Use separate sheets of paper for each problem and make sure your name is on all the pages.

1. Let X and Y be topological spaces and let $f : X \to Y$ be a path. Define a covering space S for Y. Suppose that $f : X \to Y$ is a continuous function that is a path, define a lifting $\tilde{f}$ of the function to the covering space. Prove that the function $f : X \to Y$ has a lifting $\tilde{f} : X \to S$ to the covering space.

2. Determine the fundamental group of the following spaces:

   (a) The letter A.

   (b) The union of a solid ball $B = \{(x, y, z) | x^2 + y^2 + z_2 \leq 1\}$ and an arc A so that $A \cap B$ is the endpoints of A.

   (c) The solid torus with a solid ball removed from the interior.

   (d) The surface of the torus.

3. Define *compact* and *Hausdorff space*. Prove that a compact Hausdorff space is Normal.

4. Define *paracompact* and *regular*. Prove that a paracompact space is regular.

5. Prove that if X and Y are Hausdorff spaces, X is compact and $f : X \to Y$ is a one-to-one continuous onto function, then f^{-1} is continuous. Give an example to show that if the condition of compactness is removed then f^{-1} need not be continuous.

6. Define *separable*, *a basis for a topological space* and *metric space*. Prove that a separable metric space has a countable basis.

7. Define *nowhere dense set*. Show that the closure of a nowhere dense set is nowhere dense. Prove that a compact Hausdorff space is not the union of countably many nowhere dense subsets.

8. Define *normal* and *metric space*. Prove that a metric space is normal.

9. Let X be the space whose set of points is the first uncountable ordinal ω_1 with the order topology; thus for each $x \in X$ the set $\{t \in X | t < x\}$ is countable. Show that X is a Hausdorff space, that X is not compact and that every infinite subset of X has a limit point.