Topology Preliminary Exam

Aug. 7, 2012

Solve any 10 of the following problems. **Note:** problems marked with an asterisk ($*$) are somewhat more challenging, therefore each of them counts as two.

1. Let X be an uncountable set and let

$$\mathcal{T} = \{U \subset X | \ \ U = \emptyset \ \text{ or } \ X \setminus U \ \text{ is at most countable}\}.$$

   (a) Verify that $\mathcal{T}$ satisfies the axioms of topology;

   (b) Show that $X, \mathcal{T}$ satisfies the T_1-axiom, but is not a Hausdorff space.

2. Let X denote the topological space defined in Problem 1 above. Prove that every bijection $f : X \to X$ is a homeomorphism.

3. Define the *product topology* on the set $X \times Y$, where each of X and Y is a topological space, and verify that the axioms are satisfied.

4. Prove that X is a Hausdorff space if and only if the diagonal $D = \{(x, x) \in X \times X\}$ is closed in $X \times X$.

5. Define *metric space* and the topology induced by the metric. Prove that every metric space X is normal, that is, for each pair of disjoint closed subsets A and B there exist disjoint open subsets containing A and B, respectively.

6. Define *connected space* and prove that if X is connected and $f : X \to Y$ is a continuous surjection, then Y is connected.

7. ($*$) Prove that the subset of $\mathbb{R}^2$ consisting of points with either both coordinates rational or with both coordinates irrational is connected.

8. Define *compact space* and prove that if X is compact and $f : X \to Y$ is a continuous surjection, then Y is compact.

9. Sketch the proof of connectedness of the product of two connected spaces.

10. Assuming compactness of the closed interval, prove that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

11. Define *locally connected* space and prove that if X is locally connected, then every connected component of X is open.

12. (∗) Give an example of a metric on the set $\mathbb{R}^2$ under which the resulting space is connected, but not locally connected.

13. Describe an example of a continuous surjection from the unit interval $[0, 1]$ onto the square $[0, 1] \times [0, 1]$ (that is, a square-filling Peano curve).

14. (∗) Based on the existence of a square-filling Peano curve (as in Problem 13 above), prove that there exists a continuous surjection from the real line $\mathbb{R}$ onto the plane $\mathbb{R}^2$.

15. (∗) Recall that a space X has *the fixed-point property* means that every continuous function $f : X \to X$ has a *fixed point*, that is, there is a point $x_0 \in X$ with $f(x_0) = x_0$. Prove or disprove the following statement:
    Suppose X is the union of two of its closed subspaces X_1 and X_2 that have exactly one common point, and that each of these subspaces has the fixed-point property. Then X has the fixed-point property.