TOPOLOGY PRELIMINARY EXAM

7/19/14

Do any 9 of the following.

Problem 1. Suppose $f : X \to X$ is a continuous map on a Hausdorff space X. Prove that the set of fixed points of f is closed.

Problem 2. Prove that metric spaces are normal.

Problem 3. Let $q : X \to Y$ be a quotient map, and $f : Y \to Z$ an arbitrary map from Y to a space Z. Show that f is continuous if and only if $f \circ q$ is continuous.

Problem 4. Let $X_0, X_1, \ldots$ be separable spaces. Prove that $\Pi_{n\in\mathbb{N}} X_n$ is separable.

Problem 5. Let A be an uncountable subset of a separable metric space. Prove that there is some point p in A such that every neighborhood of p contains uncountably many points of A.

Problem 6. Prove that if X and Y are connected, then $X \times Y$ is connected.

Problem 7. Prove that the collection of components of a space X forms a partition of X into closed subsets.

Problem 8. Define the one-point compactification of a locally compact Hausdorff space, and prove that it is compact and Hausdorff.

Problem 9. Prove that any product of path-connected spaces is path-connected.

Problem 10. Prove that paracompact Hausdorff spaces are regular.

Problem 11. Prove that if $K_1, K_2, \ldots$ are non-empty, compact, and connected subsets of a Hausdorff space X, and $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$, then $K = \bigcap_{n=1}^{\infty} K_n$ is non-empty, compact, and connected.

Problem 12. Let X be a space. Define the product $\alpha * \beta$ of paths in X, and show that if α is path-homotopic to α' and β is path-homotopic to β', then $\alpha * \beta$ is path-homotopic to $\alpha' * \beta'$.

Problem 13. Let $f : S^1 \to Y$ be continuous. Show that f is nullhomotopic iff f extends to a continuous $\overline{f} : B^2 \to Y$, where $B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Problem 14. Let $p : E \to B$ be a covering map, with $p(e_0) = b_0$. Define $\phi : \Pi_1(B, b_0) \to p^{-1}(b_0)$ by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is a lifting of f starting at e_0. Prove that ϕ is onto if E is path-connected, and is a bijection if E is simply connected.