Preliminary Exam Algebra

Summer 2020

Read this entire exam. Attempt at least 2 problems from each part, and at least 8 problems in total. The two additional problems can come whichever part you choose.

Part 1: Group Theory

1. State the three Sylow Theorems, and prove the first.

2. Let G be a finite group. Show that the following conditions are equivalent:

   i) G is solvable.
   ii) If H is a non-trivial epimorphic image of G, then H has a non-trivial normal Abelian subgroup.

3. Let G be a finite group. Show that the following hold:

   a. If H is a normal subgroup of G and P is a p-Sylow subgroup of H, then $G = N_G(P)H$ where $N_G(P)$ denotes the normalizer of P in G.
   b. Let F(G) be the intersection of all maximal subgroups of G. Show that F(G) is a normal subgroup of G such that G is nilpotent if and only if G/F(G) is nilpotent.

4. Let G be a finite nilpotent group. Show that if m divides |G|, then G contains a subgroup of order m. Give an example that the converse fails

5. Determine the number of non-isomorphic Abelian groups of order 1250.

Part 2: Commutative Ring Theory

1. State the Fundamental Theorem for Finitely Generated Modules over a PID.

2. Show that an integral domain R is a PID if and only if all submodules of free R-modules are free.

3. Give an example of an integral domain which is not a UFD.

4. State and give an outline of a proof of Eisenstein's criterion.

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Part 3: Galois Theory

1. State the first Main Theorem of Galois Theory.

2. Give examples that show that the first Main Theorem of Galois Theory fails for a finite field extensions if $E > K$ is not separable and if $E > K$ is not normal.

3. a. Show that every field of characteristic 0 is perfect.
   b. Show that every finite field is perfect.
   c. Give an example of a field F which is not perfect.

4. Give an outline of the proof that every field K has an algebraic closure.

5. a. Find the Galois group of $f(x) = x^4 - 2$ over Q.
   b. Find the Galois group of $f(x) = x^4 + 4x^2 - 5$ over Q.