FINAL EXAM: MATH 7320 — ALGEBRA PRELIMINARY EXAM — SPRING 2021
DR. LUKE OEDING

Name (*please print*): ______________________________

	correct	partial
1		
2		
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	Final Exam	Preliminary Exam
score		

Algebra Preliminary Exam — Spring 2021

Instructions: No outside sources are allowed. Write clearly and legibly. Start each problem on a new page and clearly label your pages if a proof continues on a separate page. You may quote theorems from the book where appropriate, and while you don't need to recall the number, you should write the statement of the theorem if you use it. Turn in the exam using adobe scan and upload to Canvas. Questions 4-11 are the final exam for 7320, Questions 1-11 are the preliminary exam. Expectation for passing the preliminary exam are at least 6-7 mostly completely correct problems.

Question List

Groups

Exercise 1. State the Classification Theorem for finitely generated $\mathbb{Z}$-modules.

Exercise 2. Classify all groups of order 8. (Hint: 3 abelian and 2 non-abelian).

Exercise 3. Compute the order of $\mathrm{GL}_n(\mathbb{F}_p)$. Find a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$, and determine the number of Sylow p-subgroups. If you can't do this for general n, p try out the example $n = 2, p = 5$.

Rings

Exercise 4. State 2 theorems attributed to Hilbert.

Exercise 5.
(1) Prove one of the following statements: (a) PID implies UFD. (b) R a UFD implies $R[x]$ is a UFD.
(2) Give a counter-example to the (false) statement: $\mathbb{Z}[x]$ is a PID.
Note that this statement means that the converse to PID implies UFD does not hold in general.

Exercise 6. Let $R = \mathbb{C}[x, y]$ and $S = \mathbb{C}[\partial_x, \partial_y]$. Compute the annihilator of $f(x, y) = 2x^2 + 11xy - 21y^2$ in S, that is, the ideal of all forms in S that annihilate f. It suffices to give a generating set of this ideal.

Modules

Exercise 7. Suppose R is a commutative ring with unity. Suppose U and V are R-modules. Recall that the *tensor product* of U and V over R, denoted $U \otimes_R V$ is the R-module generated by all formal symbols $u \otimes v$ (for $u \in U$ and $v \in V$) such that for all $u, u' \in U,\ v, v' \in V,\ r \in R$: (i) $(u + u') \otimes v = u \otimes v + u' \otimes v$, (ii) $u \otimes (v + v') = u \otimes v + u \otimes v'$, (iii) $(ru) \otimes v = u \otimes (rv)$.
Solve the following:
(1) Prove that if U and V are free R-modules, then $U \otimes_R V$ is a free R-module. (*Try the finite rank case first.*)
(2) Prove that if U and V are finitely generated R-modules respectively presented by R-matrices A and B, then $U \otimes_R V$ is an R-module presented by the Kronecker product $A \otimes B$ (unfortunately we use the same symbol).

Exercise 8.
(1) Find the canonical diagonal form of the matrix $M = \begin{pmatrix} x & -4 & -2 \\ 1 & x+4 & 1 \\ 0 & 0 & x+2 \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{Q}[x])$.
(2) Is the cokernel of M a free $R = \mathbb{Q}[x]$-module? Why or why not?
(3) Explain why the image and kernel of M are free $R = \mathbb{Q}[x]$-modules, and determine their ranks.
(*Recall the image of a map defined by a matrix is the R-linear span of the columns.*)

Representation Theory

Exercise 9. State and prove Schur's Lemma for complex irreducible representations of a finite group.

Exercise 10. Prove that the sign representation and the trivial representation are the only one-dimensional representations of the symmetric group.

Exercise 11. Consider $R = \mathbb{C}[x_1, x_2, x_3]$ as a module induced by the action of $\mathfrak{S}_3$ on $\{x_1, x_2, x_3\}$ that permutes the variables. Let R_3 denote the homogeneous polynomials of degree 3. Find all irreducible $\mathfrak{S}_3$-submodules of R_3.

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