Algebra Preliminary Exam

Spring 2022

Instructions: Do all 9 problems. Each problem is worth 20 points. Only the final 6 problems will count toward your grade for the final exam for Math 7320.

Group/Ring Theory

1. This problem has two parts.

(a) (10 points) State the Fundamental Theorem of Finitely Generated Abelian Groups, and compute, with justification, the number of abelian groups of order 72 (up to isomorphism).

(b) (10 points) Prove that every nonzero prime ideal in a PID is maximal.

2. Let R and S be commutative rings, $f : R \to S$ a ring homomorphism, and $P \subseteq R$ a prime ideal. Let $f(P)$ denote the image of P under f.

(a) (8 points) Prove that, if f is surjective and $\ker(f) \subseteq P$, then $f(P)$ is prime.

(b) (6 points) Show that (a) is false when the assumption that f is surjective is removed.

(c) (6 points) Show that (a) is false when the assumption that $\ker(f) \subseteq P$ is removed.

3. This problem has two parts.

(a) (10 points) Prove that there is a ring isomorphism $\mathbb{C}[x]/(x^2 + 1) \cong \mathbb{C}[x]/(x^2 - 1)$.

(b) (10 points) Prove that (a) is false if $\mathbb{C}$ is replaced with $\mathbb{R}$.

Linear Algebra and Modules

4. (20 points) Compute the Jordan canonical form of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}.$$

over $\mathbb{C}$. You may use, without justification, that the characteristic polynomial of A is $(x+1)^4$.

5. (20 points) Recall that a commutative ring R is called *local* if it contains exactly one maximal ideal. Let M and N be finitely generated modules over a local ring R. Prove that $M \otimes_R N = 0$ if and only if $M = 0$ or $N = 0$.

6. Let A be a square matrix over a field F, and let A^T denote its transpose. Prove that A is similar to A^T via the following steps.

(a) (10 points) Prove that A is similar to A^T when A is a Jordan block.

(b) (6 points) Prove that A is similar to A^T when the characteristic polynomial of A splits completely over F.

(c) (4 points) Prove that A is similar to A^T in general.

Fields and Galois Theory

7. (20 points) Let K/F be a field extension, and let $\alpha \in K$ be algebraic over F. Recall that the *minimal polynomial of* α is defined to be the monic polynomial $f \in F[x]$ of smallest degree such that $f(\alpha) = 0$. Prove that the minimal polynomial f of α is irreducible.

8. Let $f = x^4 - 7 \in \mathbb{Q}[x]$.

(a) (5 points) Prove that f is irreducible.

(b) (15 points) Let K denote the splitting field of f over $\mathbb{Q}$. Compute $[K : \mathbb{Q}]$.

9. (20 points) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Write down all intermediate fields of the extension $K/\mathbb{Q}$, being sure to fully justify your answer.