FINAL EXAM MATH 7320 AND PART 1 OF THE ALGEBRA PRELIMINARY EXAM SPRING 2025, DR. LUKE OEDING

Instructions: Closed notes, closed book, no electronic devices, no collaboration. Write each answer starting on a new sheet of paper, and label all pages with your name and problem number.

Complete and turn in this part of the exam in 2 hours. For the preliminary exam, you may obtain part 2 after turning in part 1, and then take the remaining time to complete it.

1. Representation Theory, and Linear Algebra over a ring

Theory

Exercise 1 (*10 points*)**.** State the Hilbert Basis Theorem.

Exercise 2 (*10 points*)**.** State the Classification Theorem for linear representations of finite groups.

Exercise 3 (*10 points*)**.** State the Classification Theorem for finitely generated $\mathbb{Z}$-modules.

Computations

Exercise 4 (*10 points*)**.** Find the complete list of isomorphism classes of abelian groups of order 2025.

Exercise 5 (*10 points*)**.** Suppose G is an abelian group presented by the matrix $\begin{pmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & -2 \end{pmatrix}$. Find the isomorphism class of G.

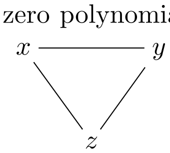
Proofs

Exercise 6 (*10 points*)**.** A module is called simple if it is not the zero module and it has no proper submodule. Prove the following version of Shur's lemma: let $\varphi : S \to S'$ be a homomorphism of simple modules. Then φ is either zero or an isomorphism.

Exercise 7 (*10 points*)**.** Prove that all finite-dimensional irreducible representations of an abelian group are 1-dimensional.

Combo

Exercise 8. Consider the polynomial ring $S = k[x, y, z]$ field k of characteristic 0 and denote by S_2 the homogeneous polynomials of degree 2 (and the zero polynomial). Let the dihedral group $\mathfrak{D}_3$ act on S_2 by the permutations obtained by $\mathfrak{D}_3$ acting on the labelled triangle



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(a) [(*15 points*)] Find the character table of the dihedral group $\mathfrak{D}_3$.
(b) [(*15 points*)] Find a decomposition of S_2 into irreducibles obtained from this group action.

Algebra Preliminary Exam (part 2)

Spring 2025

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Groups, rings, and modules.

1. Let G be a group, and let $Z(G)$ denote the *center* of G, i.e.

$$Z(G) := \{z \in G : zg = gz \text{ for all } g \in G\}.$$

(a) (5 points) Prove that $Z(G)$ is a normal subgroup of G.

(b) (15 points) Prove that, if $G/Z(G)$ is cyclic, then G is abelian.

2. (a) (10 points) Let k be a field. Prove that the rings $k[x, y]/(x^2 - y)$ and $k[x, y]/(x^2 - y^2)$ are not isomorphic.

(b) (10 points) Prove that $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x, y]$.

3. Let R be a principal ideal domain.

(a) (10 points) Suppose $P \subseteq R$ is a nonzero prime ideal. Prove that P is maximal.

(b) (10 points) Suppose $p \in R$ is a prime element. Prove that p is irreducible.

4. Let R be a commutative ring, $I \subseteq R$ an ideal, and M an R-module.

(a) (5 points) Show $\operatorname{Ann}_I(M) := \{m \in M : rm = 0 \text{ for all } r \in I\}$ is a submodule of M.

(b) (15 points) Prove that there is an isomorphism $\operatorname{Hom}_R(R/I, M) \cong \operatorname{Ann}_I(M)$.

5. (a) (15 points) Let R be a commutative ring, and let M be an R-module. Use the universal property of the tensor product to prove there is an isomorphism $M \otimes_R R \cong M$.

(b) (5 points) Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic $\mathbb{R}$-vector spaces.