Preliminary Examination in Analysis, 1:00-4:00 pm, August 17, 2018

Name:______________________________

(Total 150 points, 75 points are required for passing)

1. Let X be a nonempty set, $\mathcal{M}$ a nonempty collection of subsets of X, and μ a mapping from $\mathcal{M}$ into $[0,\infty]$.

(a) **(6 points)** Define what it means to say that $\mathcal{M}$ is an algebra on X and what it means to say that $\mathcal{M}$ is a σ-algebra on X. Further, assuming that $\mathcal{M}$ is a σ-algebra, define what it means to say that μ is a measure on $\mathcal{M}$.

(b) **(6 point)** Assume that $\mathcal{M}$ is an σ-algebra on X and μ is a measure on $\mathcal{M}$. If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, prove that $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.

2. Let X be a nonempty set and $\mathcal{P}(X)$ the collection of all the subsets of X. Let $\mathcal{A}_0 \subset \mathcal{P}(X)$ be an algebra on X, and $\mu_0 : \mathcal{A}_0 \to [0,\infty]$ be a premeasure on $\mathcal{A}_0$ (i.e. $\mu_0(\emptyset) = 0$ and if $\{A_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{A}_0$ and $\cup_{j=1}^{\infty} A_j \in \mathcal{A}_0$, then $\mu_0(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu_0(A_j)$).

(a) **(6 points)** Define what it means to say that $\mu^* : \mathcal{P}(X) \to [0,\infty]$ is an outer measure on X and what it means to say that a set $A \subset X$ is μ^*-measurable. What is the outer measure induced from μ_0?

(b) **(8 points)** Assume that μ^* is the outer measure induced from μ_0. Prove that if $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$, where $\mathcal{A}_{\sigma\delta}$ is the collection of countable intersections of sets in $\mathcal{A}_\sigma$ and $\mathcal{A}_\sigma$ is the collection of countable unions of sets in $\mathcal{A}_0$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\bar{\mathbb{R}} = [-\infty, \infty]$.

(a) **(6 points)** Define what it means to say that a function $f : X \to \bar{\mathbb{R}}$ is measurable. Assuming that $f : X \to [0,\infty]$ and $g : X \to \bar{\mathbb{R}}$ are measurable, what is the integral of f on X (with respect to μ) and what does it mean that g is integrable (with respect to μ)?

(b) **(8 points)** If $\{f_n\}$ is a sequence of measurable functions on X, prove that the set $\{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set (it is said that $\lim_{n\to\infty} f_n(x)$ exists if $-\infty < \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) < \infty$).

(c) **(6 points)** If $f : X \to [0,\infty]$ is measurable, let $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Prove that λ is a measure on $\mathcal{M}$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) **(6 points)** State the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem.

(b) **(6 points)** Use the Monotone Convergence Theorem to prove Fatou's Lemma.

(c) **(6 points)** Compute the following limit and justify your calculations, $\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} dx$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) **(4 points)** State the definition of the space $L^p(X, \mathcal{M}, \mu)$ and its norm $\|\cdot\|_p$ for $1 \le p < \infty$ and state the definition of the space $L^\infty(X, \mathcal{M}, \mu)$ and its norm $\|\cdot\|_\infty$.

(b) **(8 points)** If $f \in L^p(X, \mathcal{M}, \mu) \cap L^\infty(X, \mathcal{M}, \mu)$ for some $1 \le p < \infty$, prove that $f \in L^q(X, \mathcal{M}, \mu)$ for any $q > p$ and $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f,\ f_1,\ f_2,\ \cdots \in L^p(X, \mathcal{M}, \mu)$ $(1 \le p < \infty)$.

(a) **(6 points)** Define what it means to say that i) f_n converges to f in measure μ, ii) f_n converges to f in $L^p(X, \mathcal{M}, \mu)$, iii) f_n converges to f weakly in $L^p(X, \mathcal{M}, \mu)$.

(b) **(8 points)** If $|f_n| \le g \in L^p(X, \mathcal{M}, \mu)$ and f_n converges to f in measure, prove that f_n converges to f in $L^p(X, \mathcal{M}, \mu)$ (Hint: You can use the conclusion that if f_n converges to f in measure μ, then there is a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges to f almost everywhere with respect to μ).

(c) **(4 points)** Give an example which shows that there are $f_n, f \in L^2(X, \mathcal{M}, \mu)$ $(n = 1, 2, \cdots)$ such that $f_n \to f$ weakly as $n \to \infty$, but $f_n \not\to f$ a.e.

7. Let $(X, \mathcal{M})$ be a measurable space

(a) **(4 points)** Define what it means to say that $\nu : \mathcal{M} \to [-\infty, \infty]$ is a signed measure on $\mathcal{M}$ and state the Lebesgue-Radon-Nikodym Theorem.

(b) **(10 points)** Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure, and $\mu =$ counting measure on $\mathcal{M}$. Prove i) $m \ll \mu$ but $dm \ne f d\mu$ for any f; ii) μ has no Lebesgue decomposition with respect to m.

8. Let $-\infty < a < b < \infty$.

(a) **(8 points)** Let $L^1([a, b])$ be the space of Lebesgue integrable functions on the interval $[a, b]$ with the Lebesgue measure and ϕ be a bounded linear functional on $L^1([a, b])$. Define the function by $g(x) = \phi(\chi_{[a,x]})$ for $x \in [a, b]$. Prove that g is absolutely continuous on the interval $[a, b]$.

(b) **(8 points)** If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, prove that f is of bounded variation on $[a, b]$.

9. Let X be a normed space on $\mathbb{R}$.

(a) **(8 points)** Prove that if X is a Banach space and X^* is separable, then X is separable.

(b) **(6 points)** If X is an infinite-dimensional Hilbert space, prove that every orthonormal sequence in X converges weakly to 0.

10. (a) **(6 point)** State the Urysohn Lemma and the Arzelá-Ascoli Theorem (for a family of continuous functions on a compact metric space X).

(b) **(6 points)** Let $K \in C([0, 1] \times [0, 1])$. For $f \in C([0, 1])$, let $Tf(x) = \int_0^1 K(x, y)f(y)dy$. Prove that $Tf \in C([0, 1])$ for any $f \in C([0, 1])$, and that $\{Tf \mid f \in C([0, 1]), \|f\|_u = \sup_{x\in[0,1]} |f(x)| \leq 1\}$ is precompact in $C([0, 1])$.