Prelim in Applied Mathematics

Saturday, June 14 2008, 9:00am - 12:00pm

Instructions. Work any 6 of the following 8 problems. You may use the textbook *Applied Mathematics* by David Logan and a calculator.

1. Show using dimensional analysis that the solution of the problem

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0,$$

with

$$u(x,0) = 0, \quad u(0,t) = u_0 > 0,$$

which corresponds to the instantaneous heating of a cold half-space from its boundary at $x = 0$, is of the form $u = u_0 F\left(\frac{x}{\sqrt{\kappa t}}\right)$.

2. Determine the leading-order behavior of the roots of the equation

$$\epsilon x^3 + x^2 + x - 2 = 0$$

for $0 < \epsilon << 1$.

3. Use singular perturbation methods to obtain a uniform approximate solution to the boundary value problem

$$\epsilon y'' + (x+1)y' + y = 0, \quad y(0) = 0, \quad y(1) = 1,$$

for $0 < \epsilon << 1$.

4. Which curve minimizes the integral

$$\int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right) dx$$

when the values of y are not specified at the end points?

5. Consider the following problem.

Find the shortest curve in the xy-plane that joins the two given points $(0, a)$ *and* $(1, b)$ *and that has a given area* A *below it (above the x-axis and between* $x = 0$ *and* $x = 1$*);* a *and* b *are positive.*

Determine a differential equation and appropriate conditions that the solution of this problem must satisfy. You don't need to solve the differential equation.

6. Discuss the solvability of the following integral equation, and solve if possible.

$$u(x) = f(x) + \frac{1}{2}\int_0^1 xtu(t)dt$$

7. The host-parasite model formulated by Leslie and Gower in 1960 has the following form:

$$H_{t+1} = \frac{\alpha H_t}{1+\gamma P_t}$$
$$P_{t+1} = \frac{\beta P_t}{1+\delta P_t/H_t}$$

where $H_0 > 0$, $P_0 > 0$, and all the parameters α, β, γ, δ are positive. Under which conditions on the parameters does there exist a unique positive equilibrium? Is this equilibrium stable?

8. Solve

$$u_{tt} = c^2 u_{xx} + e^t \sin(5x) \quad 0 < x < \pi \quad t > 0$$

with $u(0,t) = u(\pi,t) = 0$, $u(x,0) = 0$, and $u_t(x,0) = \sin(3x)$.