Name: August 15, 2012

Preliminary Doctoral Examination in Applied Mathematics

Work any 6 of the following 8 problems. You may use the textbook *Introduction to the Foundations of Applied Mathematics* by Mark Holmes. If you work more than six then clearly indicate which of the six problems you would like to have graded. If you fail to indicate which six problems should be graded, then they will be chosen by the grader. Each problem is worth 10 points. You may work a seventh problem for a five-point bonus. If you do, please indicate clearly which is your bonus problem.

1. A piece of shrapnel of density ρ is driven off an explosive device at velocity v. The density of the explosive is ρ_e, and E is its Gurney energy (joules/kilogram), or the specific energy available in the explosive to do work. Determine how the velocity of the shrapnel depends upon E.

2. Find a 2-term perturbation solution of

$$u' + u = \frac{1}{1+\epsilon u}, \quad u(0) = 0, \quad 0 < \epsilon \ll 1.$$

3. Determine the leading order behavior of the three roots of $\epsilon x^3 - x - 2 = 0$ for $0 < \epsilon \ll 1$.

4. Find a composite expansion of

$$\epsilon y'' + y' + y^2 = 0, \quad y(0) = \frac{1}{4}, \quad y(1) = \frac{1}{2},$$

where $0 < \epsilon \ll 1$.

5. An allosteric enzyme E reacts with a substrate S to produce a product P according to the mechanism

$$S + E \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C_1 \overset{k_3}{\rightarrow} E + P$$

$$S + C_1 \underset{k_5}{\overset{k_4}{\rightleftharpoons}} C_2 \overset{k_6}{\rightarrow} C_1 + P$$

where the k's are rate constants and C_1 and C_2 enzyme-substrate complexes. With lower case letters denoting concentrations, and initial conditions

$$s(0) = s_0, \quad e(0) = e_0, \quad c_1(0) = c_2(0) = p(0) = 0.$$

a) Write down the differential equation model based on the *Law of Mass Action.*

b) Find the conservation law(s) for these reactions.

6. Determine the stability of the steady states of the system

$$\frac{dx}{dt} = a - bx - y^2, \quad \frac{dy}{dt} = xy - by$$

where a and b are positive constants and $a - b^2 > 0$.

7. The kinetic energy for a regular region $R(t)$ is

$$K = \iiint\limits_{R(t)} \frac{1}{2}\rho v \cdot v \, dV.$$

Let K_0 be the value of K when the motion is irrotational, which means that the velocity can be written as $v = \nabla\phi$. Let v be any other velocity, not necessarily irrotational, but which has the same normal velocity at the boundary as the irrotational motion. This means that $v \cdot n = (\nabla\phi) \cdot n$ on ∂R.

Assuming the density is constant, show that $K_0 \leq K$. This observation that irrotational flows minimize the kinetic energy is known as Kelvin's Minimum Energy Theorem. (Hint: the second identity on p. 452 in the text from Vector Calculus and the Divergence Theorem may be helpful here)

8. An incompressible viscous fluid of constant density under no body forces is confined to lie between two stationary infinite planes $z = -d$ and $z = d$. A flow is forced by a constant pressure gradient $\nabla p = [-C, 0, 0]^T$, $C > 0$. Determine the fluid velocity v.