Graph Theory Prelim – 2005

1. Let G be a simple graph. In each of the following cases, find the number of spanning trees G has (some parts may have more than one answer – if so, discuss each possibility):
    a. G contains exactly two cycles, one of length x and one of length y.
    b. G contains an edge whose removal leaves a graph with 2 components, one having i spanning trees, the other having j spanning trees.

2. Let G be a simple graph. Find necessary and sufficient conditions for G to have a (not necessarily proper) 2-edge-coloring such that each vertex is incident with at least one edge of each color. Prove your answer is true. (Hint: There are several ways to approach this, but it may help to add a vertex to G and join it to each vertex of odd degree in G.)

3. All graphs in the following are simple.
    a. State Hall's Theorem, preferably as a theorem about matchings in bipartite graphs.
    b. Suppose that G is a bipartite graph with bipartition {X, Y} of the vertex set. Suppose that every vertex in X has degree d_1, and every vertex in Y has degree d_2. Suppose that $0 < d_2 \leq d_1$. Prove that there is a matching in G which saturates X.
    c. Prove that every regular bipartite graph has edge-chromatic number equal to its degree.

4. Let G be a finite simple graph with n vertices. The *coloring number* of G is the smallest integer k for which there exists an ordering $v(1),...,v(n)$ of the vertices of G such for each i, $v(i)$ is adjacent to at most k - 1 of the vertices $v(j)$ with $j < i$.
    a. Show that the chromatic number of G is no greater than the coloring number of G.
    b. Give an example of a finite simple graph in which the coloring number and the chromatic number are equal, and an example in which they are not equal.

5. Let G be a finite simple graph G with minimum degree $\delta(G) \geq 2$.
    a. Show that G contains a set of $\delta(G)$ -1 cycles of different lengths. [Hint: think of a vertex at an end of a longest path in G.]
    b. Suppose that G contains a maximal path of length exactly $2\delta(G)$ -1. Show that G must contain a cycle of length $2\delta(G)$.