Graph Theory Prelim – 2007

1. Suppose G_1 is a 3-regular connected simple graph on *n* vertices.
    a. Find an example of G_1 that has a cut-edge and a 1-factor.
    b. Find an example of G_1 that has a cut-edge and no 1-factor. (Hint: Tutte's Theorem with $|S| = 1$ may help you see the structure of such a graph.)
    c. Suppose that G_1 has a partition of its edges into sets of size 3, each element of which induces a path of length 3.
        i. In terms of *n*, how many paths are induced by this partition?
        ii. Show that G_1 has a 1-factor. (Hint: use (i), and consider choosing the middle edge in each path.)

2. Suppose G_2 is a 3-regular connected simple graph on *n* vertices that has a 1-factor.
    a. Find an example of G_2 for which $\chi'(G_2) = 4$.
    b. Show that G_2 has a 2-factor.
    c. Show that G_2 has a partition of its edges into sets of size 3, each element of which induces a path of length 3. (Hint: Direct the edges in a suitable 2-factor to form directed cycles, and heed the suggestion in (1c(ii)).)

3. Let G_3 be a $2x$-regular simple graph.
    a. Does G_3 necessarily have an Euler tour? Why or why not?
    b. Show that the edges of G_3 can be directed so that at each vertex *v* in the resulting directed graph D_3, $d^+(v) = d^-(v)$.
    c. Form a bipartite graph B_3 on the vertex set $V(G_3) \times \{1,2\}$ by joining $(v,1)$ to $(w,2)$ if and only if there is an edge in D_3 directed from *v* to *w*.
        i. Show that B_3 has a 1-factorization.
        ii. Use this to show that G_3 has a 2-factorization.

4. A subgraph of a graph is said to be an *odd factor* if it is both spanning and all its vertices have odd degree.
    a. By counting the number of edges in G in terms of the degrees of its vertices, show that the number of vertices of odd degree in G is even.
    b. Let T be a tree with an even number of vertices. If *e* is an edge in T then *e* is said to be an *even* edge if when deleted the two remaining components each have an even number of vertices, and is said to be an *odd* edge otherwise. Show that:
        i. Every odd factor of T must contain every odd edge of T, and
        ii. Every odd factor of T contains no even edges of T.

        (Hint: Use 4a.)
    c. Find necessary and sufficient conditions in terms of *n* for a tree T tree on *n* vertices to have an odd factor.