# Graph Theory Prelim – 2011

1. Recall that a set S of vertices of a graph G is *independent* if no edge of G has both of its ends in S.

   a. Find a loopless graph on 50 vertices and 392 edges having no independent set of size 4.

   b. Prove that every loopless graph on 50 vertices and 391 edges has an independent set of size 4.

   (Hint: every simple graph has a complement.)

2. A vertex coloring of a finite simple graph G is said to be *greedy* if it is a coloring with the positive integers arising from an ordering $v_1, \ldots, v_n$ of the vertices in V(G) by following the rule:

   $v_1$ is colored 1; then, for each $j > 1$ (assuming all $v_i$ have been colored for $i < j$) $v_j$ is colored with the smallest positive integer that is not the color of a neighbor of $v_j$ among $v_1, \ldots, v_{j-1}$.

   If $f$ is a coloring of V(G) with positive integers then let $k(f) = \max_{v \text{ in } V(G)}\{f(v)\}$. Let $C_j = f^{-1}(\{j\})$ for $1 \le j \le k(f)$.

   a. Show that if $f$ is a greedy coloring then $k(f) \le \Delta(G) + 1$.

   b. Show that $f$ is a greedy coloring of G if and only if

      i. $C_j$ is an independent set of vertices, and

      ii. For $j > 1$, each vertex in $C_j$ has a neighbor in $C_i$ for $1 \le i < j$.

   c. Show that each finite simple graph G has a greedy coloring $f$ such that $k(f) = \chi(G)$.

3. Let $k$ be a positive integer, and let $V = \{0, 1, 2, \ldots, 2k\}$, and define a function $d$ on V by

   $$d(i) = i+1 \text{ if } i < k, \quad \text{and } d(i) = i \text{ if } i \ge k.$$

   Prove that there is exactly one simple graph on vertex set V and degree function $d$. Find the graph when $k = 4$.

4. Hall's Theorem can be stated in various ways. Here are two.

   a. Let $A_1, \ldots, A_n$ be finite sets. There exist elements $a_1, \ldots, a_n$ such that

      i. $a_i$ is in $A_i$ for $1 \le i \le n$, and

      ii. $a_1, \ldots, a_n$ are distinct (no two are equal)

      if and only if each subset J of $\{1, \ldots, n\}$ satisfies $|\cup_{j \text{ in } J} A_j| \ge |J|$.

   b. If B is a finite bipartite graph with bipartition {X,Y} of the vertex set, then there is a matching in B saturating the vertices in X if and only if each subset S of X satisfies $|S| \le |N_B(S)|$.

   Prove that the second version of Hall's Theorem follows from the first.

   Hall noticed that the first version could be strengthened as follows.

   c. Let $A_1, \ldots, A_n$ be finite sets and $k_1, \ldots k_n$ be positive integers. There exist sets $B_1, \ldots, B_n$ such that

      (1) $B_i$ is a subset of $A_i$ for $1 \le i \le n$, (2) $B_1, \ldots, B_n$ are pairwise disjoint, and (3) $|B_i| = k_i$ for $1 \le i \le n$

      if and only if each subset J of $\{1, \ldots, n\}$ satisfies $|\cup_{j \text{ in } J} A_j| \ge \Sigma_{j \text{ in } J}\, k_j$.

   Find an equivalent theorem to this strengthened result (4c) that is in terms of bipartite graphs in the same way that (4b) is equivalent to (4a).