Matrix Theory Exam
June 2012

Please start each numbered problem on a new sheet of paper.

(1) Eigenvalue basics

(a) State the definition of algebraic multiplicity and geometric multiplicity. Also state the inequality between these two multiplicities.

(b) Prove the inequality from the previous question.

(2) Normal Matrices and Schur's Theorem.

(a) State Schur's theorem.

(b) State the definition of normal matrix. State the characterization (an if and only if statement) of normal matrices related to Schur's Theorem.

(c) Prove Schur's Theorem or the characterization of normal matrices you stated, your choice.

(3) Eigenvalues of Hermitian Matrices.

(a) State the Courant Fischer Theorem.

(b) Let A be Hermitian, and let B be the matrix obtained by removing the 1^{st} row and column from A. State the inequalities that exists between the eigenvalues of A and B. What is the term for these inequalities?

(c) Let A be Hermitian. State the inequalities that exists between the eigenvalues of A and the diagonal entries of A. What is the term for these inequalities?

(d) Prove, your choice, (a) from basic principles, (b) from basic principles or as a consquence of (a), or (c) from basic principles or as a consquence of (b).

(4) Matrices and Polynomials.

(a) Given a polynomial with complex coefficients $f(t)$ and a complex $n \times n$ matrix A, what can be said about the eigenvalues of the matrix $f(A)$?

(b) State the Cayley Hamilton theorem.

(c) Let $f(t) = t^2 - 3t + 2$ and $g(t) = t^2 - 5t + 6$. Suppose A square matrix and $f(A)$ is singular, while $g(A)$ is invertible. What can be concluded about the eigenvalues of A?

(d) Prove either the Cayley Hamilton theorem or the statement you give in part (a), your choice.

(5) Similarity

(a) Let $A, B \in C^{n \times n}$. State the definition of A is similar to B.

(b) Show that if A and B are similar, then they have the same eigenvalues with the same algebraic and geometric multiplicities.

(c) Give an example showing that the converse of the previous statement is false.

(d) Give a neccesary and sufficient condition for a square matrix A to be similar to a diagonal matrix.

(6) More similarity.

(a) State the Jordan Canonical Form Theorem (JCF) for $A \in C^{n\times n}$. Include description of the basic Jordan block $J_k(\lambda)$.

(b) State the Rational Canonical Form Theorem (RCF) for $A \in \mathrm{F}^{n\times n}$ where F is an arbitrary field. Include description of the companion matrix $C(p)$ of the monic polynomial $p(t)$ with coefficients in F.

(c) Find the JCF of $C(t^2(t-1)) \oplus C(t^3(t-1)^2)$

(d) Find the RCF of $J_3(1) \oplus J_2(1) \oplus J_2(0)$

(e) Suppose that $N \in C^{5\times 5}$ is nilpotent, $N^2 \neq 0$ and $N^3 = 0$. Is this enough information to determine the JCF? If so, give the JCF, if not list possible JCF.

(f) Suppose that $N \in C^{6\times 6}$ is nilpotent, $\mathrm{rank}(N) = 3, \mathrm{rank}(N^2) = 1$ and $N^3 = 0$. Is this enough information to determine the JCF? If so, give the JCF, if not list possible JCF.

(7) Nonnegative matrices.

(a) State Perron's Theorem for positive matrices.

(b) Evaluate $\lim_{k\to\infty} A^k$, where $A = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

(c) Suppose that A is nonnegative $n \times n$ and A^k is positive for some positive integer k. Use Perron's Theorem to show that A has a positive eigenvalue ρ with algebraic multiplicity 1 and $\rho > |\lambda|$ for every eigenvalue λ of A which is not equal to ρ. What is the term used to describe matrices satisfying this condition?

(d) State the definition of irreducible matrix. Suppose that A is square, nonnegative, irreducible and has k eigenvalues of maximum modulus. List several characterizations of the integer k.

(e) Let $A = \begin{bmatrix} 0 & e^T & 0 \\ 0 & 0 & e \\ 1 & 0 & 0 \end{bmatrix}$ where e is the column of n ones. Find the eigenvalues, with algebraic multiplicities, of A