The Prelim Exam in Modern Stochastic Processes

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Show all of your work. Work on each problem 1,2,... on separate sheets of paper.
The passing level is 66.66%.

1. Let N_t be a standard Poisson process with intensity λ and arrival times S_n. Let θ_k be i.i.d. uniform U[0,1] random variables. Define the point process on $\mathbb{R} \times [0, \infty)$ by counting the number of random points in a Borel set B in the plane,

$$M(B) = \#\{\,(\theta_k, \tau_k) \in B\,\}\,.$$

Compute $\mathsf{P}(M(C) = 1)$, *where* C *is the disk* $(x-a)^2 + (y-b)^2 \le r^2$ *with* $r \le a \le 1 - r$, $r \le b$. *Explain all the steps.*

2. Consider an explosion in which some elementary particles are emitted. A counter registers the occurrence of a particle by (e.g., by a "beep"). Suppose that the average number of particles decreases with the reciprocal of the distance squared to the source. That is, for some constant $\lambda > 0$,

$$EN(r) = \frac{\lambda}{r^2}$$

Assume the Poisson model. Give its description.
Define the "dangerous zone" when the number of particles exceeds level 10, and the "safe zone", when the number is less than 10. Find the probability that the region beyond the distance $r = 2$ *belongs to the safe zone.*

3. Consider a random walk on the whole numbers $0, 1, 2, \ldots$ with the left trend and reflecting screen at 0. That is, for some $p, q > 0$, $q > p$,

$$\begin{aligned}
&\mathsf{P}(0 \mapsto 1) = 1\\
&\mathsf{P}(i \mapsto i+1) = p, \quad i \ge 1,\\
&\mathsf{P}(j \mapsto j-1) = q, \quad j \ge 1,\\
&\mathsf{P}(i \mapsto j) = 0, \quad |i-j| \ge 2.
\end{aligned}$$

2.1. *Write the transition matrix.*

2.2. *Show that all states "communicate", i.e., that the Markov chain is irreducible.*
Is the process periodic or aperiodic?

2.3. *Find the stationary probabilities.*

2.4. Show that in a general periodic Markov chain *the communicable states have the same period* $d(i)$. *That is, if* $i \leftrightarrow j$, *then* $d(i) = d(j)$.

4. Consider a lifetime of a system described by a renewal process with inter-event i.i.d. random variables X_n. Suppose that each X_n itself is a sum of two exponential independent random variables with intensity λ. Put $S_n = X_1 + \cdots + X_n$. The process is terminated at a random moment N with a geometric distribution with parameter p. *Compute the average lifetime, i.e.,* $\mathsf{E}\, S_N$.

5. Formulate the Blackwell Renewal Theorem about the limit behavior of the increment $m(t+a) - m(t)$ of the renewal function (give its definition, too). Formulate the Key Renewal Theorem about the limit behavior of $\int_0^t h(t-s)\, dm(s)$. *Prove that they are equivalent, that is, they follow from each other.* You may omit the periodic case.

6. Let M_n be a martingale, $M_0 = 0$. Denote its increments by $X_n = M_n - M_{n-1}$, $n \geq 1$.

6.1. *If* $\mathsf{E}\, M_n^2 < \infty$, *show that the increments are orthogonal.*

6.2. Define $V_n = \exp\{-M_{n-1}\}$.
Prove that $Y_n = X_1 V_1 + \cdots X_n V_n$ *is a martingale, provided* $X_k V_k$ *is integrable for every* k.

6.3. *Find the Doob's decomposition of* Y_n^2.

7. Let $B(t)$ be a standard Brownian motion.

7.1. Let T_a be the first moment when $B(t)$ hits the level a. *Find the distribution of* T_a, *i.e.,* $\mathsf{P}(T_a \leq x)$.

7.2. *Find the distribution of* $|B(t)|$ *and of* $|\min_{s \leq t} B(s)|$
7.3. *Which of the following processes is also a Brownian motion?*

a) $X(t) = e^{-t} B(e^{2t}, \quad t \geq 0.$
b) $Y(t) = (1-t) B\left(\dfrac{t}{1-t}\right), \; 0 < t < 1.$
c) $Z(t) = tB(1/t)$, $t > 0$. How would you define $Z(0)$?