Preliminary Examination in Analysis
8:30 am to 12:00 pm, July 28, 2025

Name:

1. Let X be a nonempty set, $\mathcal{E}$ and $\mathcal{M}$ nonempty collection of subsets of X, and μ a mapping from $\mathcal{M}$ into $[0, \infty]$.

(a) **(6 points)** Define what it means to say that $\mathcal{M}$ is a σ-algebra on X. What is the σ-algebra generated by $\mathcal{E}$? Further, assuming that $\mathcal{M}$ is a σ-algebra, define what it means to say that μ is a measure on $\mathcal{M}$.

(b) **(8 points)** What is the Borel σ-algera, denoted by $\mathcal{B}_{\mathbb{R}}$, on $\mathbb{R}$? Prove that the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ on $\mathbb{R}$ is generated by $\mathcal{E} = \{(a, b] : a, b \in \mathbb{R},\ a < b\}$.

2. Let X be a nonempty set and $\mathcal{P}(X)$ the collection of all the subsets of X. Let $\mathcal{A}_0 \subset \mathcal{P}(X)$ be an algebra on X, and $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ be a premeasure on $\mathcal{A}_0$ (i.e. $\mu_0(\emptyset) = 0$ and if $\{A_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{A}_0$ and $\cup_{j=1}^{\infty} A_j \in \mathcal{A}_0$, then $\mu_0(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu_0(A_j)$).

(a) **(6 points)** Define what it means to say that $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure on X and what it means to say that a set $A \subset X$ is μ^*-measurable. What is the outer measure induced from μ_0?

(b) **(8 points)** Let m be the Lebesgue measure on $\mathbb{R}$. Suppose that $A \subset \mathbb{R}$ is Lebesgue measurable, $m(A) < \infty$, and for every interval I, $m(A \cap I) \leq \frac{1}{2} m(I)$. Prove that A has Lebesgue measure zero.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\bar{\mathbb{R}} = [-\infty, \infty]$.

(a) **(6 points)** State the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem.

(b) **(8 points)** If $\{f_n\}$ is a sequence of measurable functions on X, prove that the set $\{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set (it is said that $\lim_{n\to\infty} f_n(x)$ exists if $-\infty < \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) < \infty$).

4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) **(6 points)** State the definition of the space $L^p(X, \mathcal{M}, \mu)$ and its norm $\|\cdot\|_p$ for $1 \le p < \infty$, and state the definition of the space $L^\infty(X, \mathcal{M}, \mu)$ and its norm $\|\cdot\|_\infty$.

(b) **(8 points)** If $f \in L^p(X, \mathcal{M}, \mu) \cap L^\infty(X, \mathcal{M}, \mu)$ for some $1 \le p < \infty$, prove that $f \in L^q(X, \mathcal{M}, \mu)$ for any $q > p$ and $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f,\ f_1,\ f_2,\ \cdots \in L^p(X, \mathcal{M}, \mu)$ $(1 \le p < \infty)$.

(a) **(6 points)** Define what it means to say that i) f_n converges to f in measure μ, ii) f_n converges to f in $L^p(X, \mathcal{M}, \mu)$, iii) f_n converges to f weakly in $L^p(X, \mathcal{M}, \mu)$.

(b) **(8 points)** If $|f_n| \le g \in L^p(X, \mathcal{M}, \mu)$ and f_n converges to f in measure, prove that f_n converges to f in $L^p(X, \mathcal{M}, \mu)$ (Hint: You can use the fact that if f_n converges to f in measure μ, then there is a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges to f almost everywhere with respect to μ).

6. Let $(X, \mathcal{M})$ be a measurable space. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

(a) **(6 points)** Define what it means to say that $\nu : \mathcal{M} \to [-\infty, \infty]$ is a signed measure on $\mathcal{M}$ and state the Lebesgue-Radon-Nikodym Theorem. State the Lebesgue Differentiation Theorem.

(b) **(8 points)** What is the Lebesgue set L_f of f? If f is continuous at x, prove that $x \in L_f$.

7. Let $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{R}$.

(a) **(6 points)** Define what it means to say that f is of bounded variation on $[a, b]$ and that f is absolutely continuous on $[a, b]$.

(b) **(8 points)** If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, prove that f is of bounded variation on $[a, b]$.

8. (a) **(6 point)** State the Arzelá-Ascoli Theorem (for a family of continuous functions on a compact metric space X), and the Uniform Boundedness Principle.

(b) **(8 points)** If X is an infinite-dimensional Hilbert space, prove that every orthonormal sequence in X converges weakly to 0.