Real Analysis **Time:** 10:00-13:00, Aug. 9th

Prelim Exam 2022 **Classroom:** Parker Hall 250

Auburn University
Auburn, AL

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Print Full (First, Last) Name: ____________________________

Instructions:

1. Folland's textbook is allowed but lecture notes and any electronic devices are not allowed during the exam.
2. Please work out the problems in the space provided and show your answers clearly and legibly. You will be provided draft papers, which won't be graded.
3. Coverage: One problem for each chapter and the following sessions will be covered:

1.1	Introduction	15 points
1.2	σ-algebras	
1.3	Measures	
1.4	Outer measures	
1.5	Borel measures on the real line	
2.1	Measurable functions	15 points
2.2	Integration of nonnegative functions	
2.3	Integration of complex functions	
2.4	Modes of convergence	
2.5	Product measures	
2.6	The n-dimensional Lebesgue integral	
2.7	Integration in polar coordinates	
3.1	Signed measures	15 points
3.2	The Lebesgue-Radon-Nikodym theorem	
3.3	Complex measures	
3.4	Differentiation on Euclidean space	
3.5	Functions of bounded variation	
4.1	Topological spaces	15 points
4.2	Continuous maps	
4.3	Nets	
4.4	Compact spaces	
4.5	Locally compact Hausdorff spaces	
4.6	Two compactness theorems	
5.1	Normed vector spaces	20 points
5.2	Linear functionals	
5.3	The Baire category theorem and its consequences	
5.4	Topological vector spaces	
5.5	Hilbert spaces	
6.1	Basic theory of L^p spaces	20 points
6.2	The dual of L^p	
6.3	Some useful inequalities	
6.4	Distribution functions and weak L^p	
6.5	Interpolation of L^p spaces	

Mark: (out of 100) ________________

Question 1 (15 points) If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Question 2 (15 points) Suppose $\{f_n\} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Then, $\int_E f = \lim \int_E f_n$ for all $E \in \mathcal{M}$. Moreover, this need not be true if $\int f = \lim \int f_n = \infty$.

Question 3 (15 points) If E is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as (whenever the limit exists)

$$D_E(x) := \lim_{r \to 0} \frac{m\left(E \cap B(r,x)\right)}{m\left(B(r,x)\right)}.$$

1. Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.
2. Find one example of E and x such that $D_E(x)$ is a given number $\alpha \in (0,1)$.
3. Find one example of E and x such that $D_E(x)$ does not exist.

Question 4 (15 points) Let (X, ρ) be a metric space. A function $f \in C(X)$ is called Hölder continuous of exponent α with $\alpha > 0$ if the quantity

$$N_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} < \infty.$$

Show that if X is compact, then the following set is compact in $C(X)$:

$$\mathcal{F} := \{f \in C(X) : \ ||f||_u \leqslant 1 \text{ and } N_\alpha(f) \leqslant 1\}.$$

Question 5 (15 points) Let $g \in L^2([0,1] \times [0,1])$ and define $L : L^2([0,1]) \to L^2([0,1])$ by $L(f)(x) = \int_0^1 g(x,y) f(y) \mathrm{d}y$.

1. Show that $||L|| \leqslant \left(\int_0^1 \int_0^1 |g(x,y)|^2 \mathrm{d}x \mathrm{d}y\right)^{1/2}$.
2. Show that there exists an $f_0 \in L^2([0,1])$ such that $||L(f_0)||_{L^2([0,1])} = ||L||$.

Hints: For part 2, you may want to apply *Alaoglu's theorem*: If $\mathcal{X}$ is a normed vector space, the closed unit ball $B^* = \{f \in \mathcal{X}^* : ||f|| \leqslant 1\}$ in $\mathcal{X}^*$ is compact in the weak star topology.

Question 6 (20 points) If $0 < \alpha < n$, define an operator T_α on functions on $\mathbb{R}^n$ by

$$T_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{-\alpha} f(y) \mathrm{d}y.$$

Show that T_α is weak type $(1, (n-\alpha)^{-1})$ and strong type (p, r) with respect to Lebesgue measure on $\mathbb{R}^n$, where

$$1 < p < \frac{n}{\alpha} \quad \text{and} \quad \frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}.$$