Mathematical Statistics Preliminary Examination

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June 15, 2005

Time : Wednesday, June 15, 10:00 am - 1:00 pm.

I. Definitions : Define each of the following:

1. Almost sure convergence
2. Statistic
3. Sufficient statistic
4. Ancillary statistic
5. UMVUE (uniform minimum variance unbiased estimator of a parameter)

II. Problems : Work the first two and any five of the remaining eight problems.

1. State and prove the central limit theorem for i.i.d. random variables.
2. State and prove the Rao-Blackwell theorem.
3. Suppose $X_1, X_2, \ldots$ be a sequence of independent random variables such that $E(X_i) = \mu_i$ and $\mathrm{Var}(X_i) = \sigma_i^2$ for $i = 1, 2, \ldots$. For $n \geq 1$, write $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^n \mu_i$. Show that if
$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^n \sigma_i^2 = 0 ,$$
then
$$\bar{X}_n - \bar{\mu}_n \stackrel{\mathcal{P}}{\longrightarrow} 0 .$$
4. Suppose X is a discrete random variable with
$$P\{X = -1\} = p$$
and
$$P\{X = n\} = (1-p)^2 p^n , \quad n = 0, 1, 2, \ldots$$
for some p with $0 < p < 1$. Then, for a sample of size one, show that X is minimal sufficient but not complete.
5. Suppose X has a Poisson distribution with mean $\lambda > 0$. The probability mass function of X is given by
$$p(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} , \quad x = 0, 1, 2, \ldots .$$
Consider estimating the parameter $q(\lambda) = e^{-3\lambda}$ based on a sample of size one using the statistic $T(X) = (-2)^X$. Show that T is an unbiased estimator for $e^{-3\lambda}$. Argue, however, that the use of T as an estimator of $e^{-3\lambda}$ is absurd.

6. Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$ for some $\theta > 0$. Show that both $2\bar{X}$ and $[(n+1)/n]X_{(n)}$, where $X_{(n)} = \max(X_1, \ldots, X_n)$, are both unbiased estimators of θ. Which of the two estimators is "better" (i.e. smaller variance)?

7. **Prove :** Let $X_{(1)} \leq \ldots \leq X_{(n)}$ denote the order statistics of a random sample $X_1, \ldots, X_n$ from a continuous population with cdf $F(x)$ and pdf $f(x)$. Then, for $1 \leq j \leq n$ the pdf of the jth order statistic $X_{(j)}$ is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)[F(x)]^{j-1}[1-F(x)]^{n-j} \ .$$

8. Let X be a random variable such that

$$E[X^{2m}] = \frac{(2m)!}{2^m m!} \ , m = 1, 2, 3, \ldots$$

and

$$E[X^{2m-1}] = 0 \ , m = 1, 2, 3, \ldots \ .$$

Find the mgf and pdf of X.

9. Let X_1, X_2, X_3 be independent and identically distributed random variables with a common pdf

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Consider the random variables Y_1, Y_2, Y_3 defined by

$$Y_1 = \frac{X_1}{X_1 + X_2 + X_3}, \quad Y_2 = \frac{X_2}{X_1 + X_2 + X_3}, \quad Y_3 = X_1 + X_2 + X_3$$

Derive the joint and marginal pdf's of Y_1, Y_2, Y_3. Are Y_1, Y_2, Y_3 independent? If not, identify the dependent pairs.

10. Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} e^{-(x-\theta)} & x > \theta, \ -\infty < \theta < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Obtain an unbiased, consistent estimator of θ.