Mathematical Statistics Preliminary Examination

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Time : Wednesday, June 21, 9:30 am.

I. Definitions : Define each of the following:

1. Likelihood ratio test
2. Pivotal quantity
3. Power function of a test
4. Unbiased confidence set
5. Asymptotic variance

II. Problems : Solve any **five** of the eight problems below.

1. Suppose $X_1, \ldots, X_n$ are independent and identically distributed as Gamma distribution with pdf
$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right) I_{(0,\infty)}(x) ,$$
where $\alpha > 0$ is known and $\beta > 0$ is unknown.
   (a) Derive the maximum likelihood estimator, $\hat{\beta}$, of β.
   (b) Identify the exact distribution of the MLE $\hat{\beta}$.
   (c) Prove that as n goes to infinity, $\sqrt{n}(\hat{\beta} - \beta)$ follows a normal distribution.
   (d) Verify that the variance in the above asymptotic distribution can also be obtained by calculating the Fisher information.
   (e) Derive the likelihood ratio test statistic for testing $H_0 : \beta = \beta_0$.
2. Suppose $X_1, \ldots, X_n$ are iid random variables from a distribution with density
$$f(x; a, b) = \frac{1}{b} \exp\left(-\frac{x-a}{b}\right) I_{(a,\infty)}(x) ,$$
where a is unknown and b is known.
   (a) Show that $X_{(1)} = \min\{X_1, \ldots, X_n\}$ is sufficient for a and complete.
   (b) Identify the asymptotic distribution of $X_{(1)}$.
   (c) Show that the UMVUE of a is $X_{(1)} - (b/n)$.

3. Suppose $X_1, \ldots, X_n$ are iid from a distribution with density

$$f(x; \lambda) = \lambda \exp(-\lambda x) I_{(0,\infty)}(x) ,$$

where $\lambda > 0$.

   (a) Calculate $E[X_1]$ and find its UMVUE.

   (b) Find the UMVUE of λ^{-2}. Does its variance attain the Cramér-Rao lower bound?

4. Let $X_1, \ldots, X_n$ be iid $N(\mu, 1)$. Suppose we want to estimate the probability that $X_1 > 0$, i.e. the function

$$1 - \Phi(-\mu) = \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\mu)^2\right) \, dx .$$

   (a) Show that $X_1 - X_2$ and $X_1 - \bar{X}$ are independent of $\bar{X}$.

   (b) Give an unbiased estimator of $1 - \Phi(-\mu)$.

   (c) Find the UMVUE of $1 - \Phi(-\mu)$.

   (d) Find the MLE of $1 - \Phi(-\mu)$.

5. Let X have the pdf $f(x)$ where f is strictly decreasing on $[0, \infty)$. Prove that for a fixed value of α, of all intervals $[a, b]$ that satisfy $\int_a^b f(x)dx = 1 - \alpha$, the shortest interval is obtained by choosing $a = 0$ and b so that $\int_0^b f(x)dx = 1 - \alpha$.

6. Suppose $X_1, \ldots, X_n$ is a random sample from the density

$$f(x; \theta) = \theta(\theta + 1)x^{\theta-1}(1 - x) I_{(0,1)}(x) ,$$

where $\theta > 0$. Show that the method of moments estimator is inefficient.

7. Let X be a sample of size one from the Bernoulli(p) distribution. Show that X is the UMVUE of p. Show, however, that not even an unbiased estimator exists for the odds ratio $p/(1 - p)$.

8. Let $X_1, \ldots, X_n$ be a random sample from the uniform$(0, \theta)$ distribution, $\theta > 0$. Let Y be the largest order statistic. Prove that Y/θ is a pivotal quantity and show that the interval

$$\left\{\theta : y \leq \theta \leq \frac{y}{\alpha^{1/n}}\right\}$$

is the shortest $1 - \alpha$ pivotal interval.