*August 03, 2007*

# Mathematical Statistics Preliminary Examination

Statistics Group, Department of Mathematics and Statistics, Auburn University

Name: ________________________________________

1. It is a closed-book and in-class exam.
2. One page (letter size, 8.5-by-11in) cheat sheet is allowed.
3. Calculator is allowed. No laptop (or equivalent).
4. Show your work to receive full credits. *Highlight your final answer.*
5. Solve any **five** problems out of the seven problems.
6. Total points are **50**. Each question is worth **10** points.
7. If you work out more than five problems, your score is the sum of five highest points.
8. Time: **150** minutes. (9:00am–11:30am, Friday, August 03, 2007)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

1. Suppose that $X_1, \dots, X_n$ are independent random variables, and $X_i \sim N(\theta_i, \sigma^2)$, $i = 1, \dots, n$, where $\sigma^2$ is a known constant. We are interested in the hypothesis $H_0 : \theta_1 = \cdots = \theta_n = 0$.

   (a) Find a size $\alpha$ UMP test for

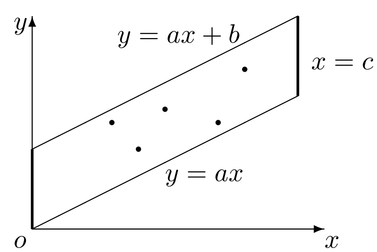
   $$H_0 :\ \theta_1 = \cdots = \theta_n = 0, \quad \text{versus} \quad H_a :\ \theta_i = \theta_{i0},\ i = 1, \dots, n.$$

   where $\theta_{10}, \dots, \theta_{n0}$ are given constants. (identify test statistic and reject region)

   (b) Find the likelihood ratio test. (identify test statistic and reject region)

   (c) Calculate the powers of the tests in (a) and (b) when the alternative hypothesis is $H_a : \theta_i = n^{-1/3}$, $i = 1, \dots, n$, and compare them as $n \to \infty$.

2. Suppose $(Y, X)$ are uniform on the area bounded by $y = ax$, $y = ax + b$, $x = 0$ and $x = c$. Suppose $(Y_1, X_1), \dots, (Y_n, X_n)$ is an iid sample from $(Y, X)$. Find the MLEs of $a$, $b$, and $c$.




3. A commuter leaves for work between 6 A.M. and 7 A.M., and takes between 1 to 2 hours to get there. Let the random variable $X$ denote her time of departure, and the random variable $Y$ the travel time. Assume that these variables are independent and uniformly distributed and let $Z = X + Y$ be the time of arrival at work.

   (a) Find the probability that the commuter arrives at work before 8 A.M.

   (b) Find the probability that the commuter arrives at work before 8 A.M. knowing that she has left home after 6:30 A.M. in a particular morning.

   (c) Find the density of $Z$, and find the conditional density of $X$ given $Z = 8.5$.

4. Suppose that $X_1, \dots, X_n$ are iid random variables with pdf

   $$f(x|\theta) = \theta x^{\theta - 1}, \quad 0 < x < 1, \quad \theta > 0.$$

   (a) Obtain the Cramér-Rao lower bound on the variance of unbiased estimators of $\theta$.

   (b) Obtain a moment estimator $\hat{\theta}$.

   (c) Investigate the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$, where $\hat{\theta}$ is the estimator found in part (b). Is it asymptotically efficient?

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution having density

$$f(x|\theta, \mu) = \frac{1}{\theta} e^{-\frac{x-\mu}{\theta}} I_{[\mu,\infty)}(x), \quad \theta > 0, \quad -\infty < \mu < \infty.$$

(a) If both $\theta$ and $\mu$ are unknown parameters, show that a level $\alpha$ likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has rejection region of the form $\sum_{i=1}^n (X_i - X_{(1)}) < C_1$ or $\sum_{i=1}^n (X_i - X_{(1)}) > C_2$, where $X_{(1)}$ is the smallest of the $X_i$'s.

(b) Now suppose that $\mu$ is known to be 0. A level $\alpha$ likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ has rejection region of the form $\sum_{i=1}^n X_i < C_1$ or $\sum_{i=1}^n X_i > C_2$. Determine explicitly values of $C_1$ and $C_2$ which yield a test of level $\alpha$. Are these values unique?

(c) Again assuming $\mu = 0$, derive a $1 - \alpha$ confidence interval for $\theta$ by inverting the test in (b).

6. Consider the family of probability density functions $\{h(z; \theta) : \theta \in \Theta\}$, where $h(z; \theta) = \frac{1}{\theta} I_{(0,\theta)}(z)$.

(a) Show that the family is complete provided that $\Theta = (0, \infty)$.

(b) Show that the family is not complete if $\Theta = (1, \infty)$.

7. Suppose $X_1, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$. Let $\tilde{X}$ be the sample median and $\bar{X}$ be the sample mean of the random sample. Show that

$$var[\tilde{X}] = var[\tilde{X} - \bar{X}] + \sigma^2/n.$$