Mathematical Statistics Preliminary Examination

Parker 250. 8:30am - 12:30pm, Monday, August 15, 2022

Name: ____

1. It is a closed-book and in-class exam.
2. One page (letter size, 8.5-by-11in) double-sided cheat sheet is allowed.
3. Electronic devices (calculator, laptop, tablet, smartphone, etc) are **not allowed**.
4. Solve **five** problems and submit solutions for **at most five** problems.
5. Start each problem on a new page. Clearly label each problem and number each page and write your name on the top right of each page.
6. Show your work to receive full credits. *Highlight your final answer.*
7. Total points are **50**. Each problem is worth **10** points.

Please circle the five problems you are submitting for grading.

1	2	3	4	5	6	7	Total

The density, mean, and variance of selected common distributions.

- Normal(μ, σ^2)

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \qquad E(X) = \mu \qquad var(X) = \sigma^2$$

- Gamma(α, β)

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \qquad E(X) = \alpha\beta \qquad var(X) = \alpha\beta^2$$

- χ^2_p

$$f(x) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2} \qquad E(X) = p \qquad var(X) = 2p$$

- Beta(α, β)

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad E(X) = \frac{\alpha}{\alpha+\beta} \qquad var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

1. Let X be an exponential random variable with density

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad x > 0, \lambda > 0.$$

Given a constant $C > 0$, we will observe T and δ as follows

$$T = \min\{X, C\}, \quad \delta = \begin{cases} 1, & \text{if } T = X, \\ 0, & \text{if } T < X. \end{cases}$$

Assume that the observations are T_i and δ_i for $i = 1, \ldots, n$.

(a) Find $\hat{\lambda}$, the MLE of λ.

(b) Find the asymptotic distribution of the MLE $\hat{\lambda}$ as $n \to \infty$.

(c) Find a 95% confidence interval of λ.

2. Let (X, Y) be uniform in a circle with radius R, that is, $X^2 + Y^2 \leq R^2$. Consider a simple random sample (X_i, Y_i) for $i = 1, 2, \ldots, n$. Let $Z_i = \sqrt{X_i^2 + Y_i^2}$.

(a) Find $\tilde{R}$, the method-of-moment estimator of R.

(b) Find the asymptotic distribution of the MM estimator $\tilde{R}$ as $n \to \infty$.

(c) Find an estimator for the area of the circle and obtain the asymptotic distribution of this estimator.

(d) Find a 95% confidence interval of the area of the circle.

3. Let $X_1, \ldots, X_n$ be iid from a distribution with density $f(x) = \theta^{-1} I(\theta \leq x \leq 2\theta)$, for $\theta > 0$.

(a) Find an unbiased estimate W as a function of $X_{(1)} = \min_i X_i$.

(b) Find an unbiased estimate T as a function of $X_{(n)} = \max_i X_i$.

(c) Which one is better, W or T? Why?

4. Assume that $X_1, \ldots, X_n$ are iid from $N(\mu, \sigma^2)$ with unknown μ and σ^2.

(a) Find the best unbiased estimator of $\theta = \mu/\sigma$.

(b) Find the mean squared error of the best unbiased estimator.

5. Let $X_1, \ldots, X_n$ be iid from a distribution with density

$$f(x) = \frac{1}{\lambda} e^{-(x-\theta)/\lambda}, \quad \lambda > 0, x > \theta.$$

(a) Find the likelihood ratio test statistic for testing $H_0 : \theta \leq \theta_0$ versus $H_a : \theta > \theta_0$.

(b) Find the rejection region of a size-α test for the above test.

6. Let $X_1, \ldots, X_n$ be iid from a distribution with density

$$f(x) = \theta^2 x e^{-\theta x}, \quad x > 0, \theta > 0.$$

(a) Find a sufficient and complete statistic T_n for the family of distributions with $\theta > 0$.

(b) Show that the distributions of T_n has monotone likelihood ratio.

(c) Find the uniformly most powerful test for $H_0 : \theta \geq \theta_0$ versus $H_a : \theta < \theta_0$ at the significance level α.

7. Assume that $X_1, \ldots, X_n$ are iid from a distribution with density

$$f(x) = \beta x^{-\beta-1}, \quad x > 1, \beta > 2.$$

(a) Construct a $100(1-\alpha)\%$ confidence interval for β.

(b) Find the asymptotic distribution of L, where L is the length of this confidence interval.