ALGEBRA PRELIMINARY EXAM, SPRING 2015

Name (*please print*): ________________________________

	total	
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
total	110	

Instructions:

- Answer each question on a new piece of paper.
- Restate each question.
- Write clearly and legibly.
- Be sure to fully explain all your answers, and give a structured, understandable argument.
- Answers will be graded on clarity and the correctness of the main steps of the reasoning.
- Though much effort has been made to eliminate typos and simple mistakes, if you notice one, ask the proctor. Do not interpret a problem in a way that would make it trivial.
- You may quote major results from the textbook (Hungerford), class notes, and homework.

Good luck!

Exercise 1. Prove that a group of order 182 is solvable. (Note: $182 = 2 \cdot 7 \cdot 13$)

Exercise 2. Suppose G is a group of order $56 = 2^3 \cdot 7$. Show that G is not simple.

Exercise 3. Classify all groups of order 2015.

Exercise 4. Find the galois group of $x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

Exercise 5. Give examples of the following objects:

(a) An irreducible polynomial over $\mathbb{Q}$ that can be proved to be so using Eisenstein's criterion for $p = 5$.
(b) A UFD that isn't a PID.
(c) A finite extension of $\mathbb{Z}_p(x)$ (the field of rational functions in x with coefficients in $\mathbb{Z}_p$) that is normal but not separable.

Exercise 6. Let $i = \sqrt{-1}$ and let x be an indeterminate. Consider $\mathbb{Z} \times \mathbb{Z},\ \mathbb{Z}[i],$ and $\mathbb{Z}[x]/\langle x^2 \rangle$.

(a) Show that all three all isomorphic as *additive groups*.
(b) Show that no two are isomorphic as *rings*.

Exercise 7.

(a) Let F and K be fields with $F \subset K$. Let $\alpha, \beta \in K$ be algebraic over F with minimal polynomials $f, g \in F[x]$. Show that f is irreducible over $F(\alpha)$ if and only if g is irreducible over $F(\beta)$.
(b) (i) Compute the factorization of $x^6 - 4$ over $\mathbb{C}$.
(ii) Let K be the splitting field of $x^6 - 4$. Compute $[K : \mathbb{Q}]$.

Exercise 8. Let F be a field and let F^* denote the nonzero elements in F. A discrete valuation on F is a function $\nu \colon F^* \to \mathbb{Z}$ such that

i $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in F^*$, i.e. ν is a homomorphism from the multiplicative group F^* to the additive group $\mathbb{Z}$.
ii ν is surjective.
iii $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for all $a, b \in F^*$ with $a + b \neq 0$.

The set $R = \{x \in F^* \mid \nu(x) \geq 0\} \cup \{0\}$ is called the *valuation ring* of ν.

(a) Prove that R is a subring of F containing the identity.
(b) Prove that for each nonzero $x \in F$, either x or x^{-1} is in R.

Exercise 9. Suppose R is a commutative ring with unity. Suppose A and B are R-modules. Recall that the *tensor product* of A and B over R, denoted $A \otimes_R B$ is the R-module generated by all formal symbols $a \otimes b$ (for $a \in A$ and $b \in B$) such that for all $a, a' \in A,\ b, b' \in B,\ r \in R$:

(i) $(a + a') \otimes b = a \otimes b + a' \otimes b$,
(ii) $a \otimes (b + b') = a \otimes b + a \otimes b'$,
(iii) $(ra) \otimes b = a \otimes (rb)$.

Prove that if A and B are projective R-modules, then $A \otimes_R B$ is a projective R-module.

Exercise 10. Suppose that $[\mathbb{Q}(u) : \mathbb{Q}]$ is odd. Show that $\mathbb{Q}(u^2) = \mathbb{Q}(u)$.

Exercise 11. Let $\mathbb{F}_2$ denote the field with 2 elements. Find an inverse of $(1 + x)^3$ in $\mathbb{F}_2[[x]]$.