**MATRICES PRELIMINARY EXAM, SUMMER 2016**

Name (*please print*): ____________________________________

Instructions:

(1) Answer each question on a new piece of paper.
(2) Restate each question.
(3) Write clearly and legibly.
(4) Be sure to fully explain all your answers, and give a structured, understandable argument.
(5) Answers will be graded on clarity and the correctness of the main steps of the reasoning.
(6) Though much effort has been made to eliminate typos and simple mistakes, if you notice one, ask the proctor. Do not interpret a problem in a way that would make it trivial.
(7) You may quote major results from the textbook.

Good luck!

**Exercise 1.** Prove 2 of the following 3 statements.

(1) Prove that matrix rank is lower semi-continuous (informally, that matrix rank can never go up in a limit).
(2) Prove that matrix rank is sub-additive, i.e. $\mathrm{rank}(A+B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$ for all $A, B \in M_n$.
(3) Suppose $A \in M_n(\mathbb{C})$ is stochastic. Prove that the rank of $A$ is at most $n-1$.

**Exercise 2.** Prove 2 of the following 3 statements.

(1) Prove that the set of real positive semidefinite matrices forms a convex cone in the Euclidean space $M_n \cong \mathbb{R}^{n^2}$.
(2) Prove that if $A \in M_n$ is Hermitian then $A$ is positive semidefinite if and only if there is a sequence of positive definite matrices $A_1, A_2, \ldots$, such that $A_k \to A$ as $k \to \infty$.
(3) Show that the determinant of the Kronecker product of two matrices $A \in M_n$ and $B \in M_m$ is

$$\det(A \otimes B) = \det(A)^m \det(B)^n$$

**Exercise 3** (H-J 1.2.P4)**.** Suppose that $A \in M_n$ is idempotent. Show that every coefficient of the characteristic polynomial $p_A(t)$ is an integer.

**Exercise 4** (H-J 1.2.P22)**.** Consider the $n \times n$ circulant matrix

$$C_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

For a given $\epsilon > 0$ let $C_n(\epsilon)$ be the matrix obtained from $C_n$ by replacing its $(n,1)$ entry by $\epsilon$. Prove the following:

(1) The characteristic polynomial of $C_n(\epsilon)$ is $p_{C_n(\epsilon)} = t^n - \epsilon$.
(2) The spectrum of $C_n(\epsilon)$ is $\sigma(C_n(\epsilon)) = \{\epsilon^{1/n} e^{2\pi i k/n} \mid k = 0, 1, \ldots, n-1\}$.
(3) The spectral radius of $I + C_n(\epsilon)$ is $\rho(I + C_n(\epsilon)) = 1 + \epsilon^{1/n}$.

**Exercise 5** (H-J 1.3.P28, 1.3.P34)**.** Prove one of the following:

(1) Let $A \in M_{m,n}$, and $B \in M_{n,m}$ be given. Prove that $\det(I_m + AB) = \det(I_n + BA)$.
(2) If $A, B \in M_n$ are similar, show that $\mathrm{adj}(A)$ and $\mathrm{adj}(B)$ are similar.

**Exercise 6** (H-J 1.4.P12)**.** Let $\lambda$ be an eigenvalue of $A \in M_n$. (a) Show that every list of $n-1$ columns of $A - \lambda I$ is linearly independent if and only if no eigenvector of $A$ associated with $\lambda$ has a zero entry. (b) If no eigenvector of $A$ associated with $\lambda$ has a zero entry, why must $\lambda$ have geometric multiplicity 1?

**Exercise 7** (H-J p90 *Cholesky factorization*)**.** Show that any $B \in M_n$ of the form $B = A^*A$, with $A \in M_n$, may be written as $B = LL^*$, in which $L \in M_n$ is lower triangular and has non-negative diagonal entries. Explain why this factorization is unique if $A$ is non-singular.

**Exercise 8** (H-J 2.1.23)**.** Let $A \in M_n$, let $A = QR$ be a $QR$ factorization, and partition $A$, $Q$, and $R$ according to their columns: $A = [a_1 \ \ldots \ a_n]$, $Q = [q_1 \ \ldots \ q_n]$, $R = [r_1 \ \ldots \ r_n]$. Explain why $|\det A| = \det R = r_{11} \cdots r_{nn}$ and why $||a_i||_2 = ||r_i||_2 \geq r_{ii}$ for each $i = 1, \ldots, n$, with equality if and only if either (a) some $a_i = 0$ or (b) $A$ has orthogonal columns (i.e. $A^*A = \mathrm{diag}(||a_1||_2^2, \ldots, ||a_n||_2^2)$).

**Exercise 9** (H-J Theorem 2.3.3)**.** Let $\mathcal{F} \subset M_n$ be a nonempty commuting family. Prove that there is a unitary matrix $U \in M_n$ such that $U^*AU$ is upper triangular for every $A \in \mathcal{F}$.

**Exercise 10.** Prove the Cayley-Hamilton Theorem: Let $p_A(t)$ be the characteristic polynomial of $A \in M_n$. Then $p_A(A) = 0$.

**Exercise 11** (H-J 2.4.P10)**.** Show that $A, B \in M_n$ have the same characteristic polynomials, and hence the same eigenvalues, if and only if $\mathrm{tr} A^k = \mathrm{tr} B^k$ for all $k = 1, \ldots, n$. Deduce that $A$ is nilpotent if and only if $\mathrm{tr} A^k = 0$ for all $k = 1, \ldots, n$.

**Exercise 12** (H-J 3.2.P16)**.** Let $A \in M_n$ have Jordan canonical form $J_{n_1(\lambda_1} \oplus \cdots \oplus J_{n_k}(\lambda_k)$. If $A$ is nonsingular, show that the Jordan canonical form of $A^2$ is $J_{n_1(\lambda_1^2}) \oplus \cdots \oplus J_{n_k}(\lambda_k^2)$. However, the Jordan canonical form of $J_m(0)^2$ is not $J_m(0^2)$ if $m \geq 2$; explain.

**Exercise 13** (H-J 4.4.P26, P28)**.** Prove one of the following

(1) Show that real matrices $A, B \in M_n(\mathbb{R})$ are complex orthogonally similar if and only if they are real orthogonally similar.
(2) Let $A \in M_n$ be given. Show that $\det(I + A\overline{A})$ is real and non-negative.

**Exercise 14.** Carefully state and prove one of the following two.

(1) Eigenvalue monotonicity for Hermitian matrices.
(2) Eigenvalue interlacing for Hermitian matrices.

**Exercise 15.** Do one of the following two.

(1) Use Jordan canonical form to explain the geometric multiplicity-algebraic multiplicity inequality.
(2) Explain the difference between a vector norm and a matrix norm for matrices. Give an example of one that is not the other.

**Exercise 16.** Let $A \in M_n$. Assume Theorem 5.6.12: $\lim_{k\to\infty} A^k = 0$ if and only if $\rho(A) < 1$.
Prove Corollary 5.6.14 (Gelfand's formula): Let $|||\cdot|||$ be a matrix norm on $M_n$. Then $\rho(A) = \lim_{k\to\infty} |||A^k|||^{1/k}$.

**Exercise 17** (H-J 5.6.P47)**.** If $A, B \in M_n$, $A$ is nonsingular, and $B$ is singular, show that

$$|||A - B||| \geq 1/|||A^{-1}|||.$$

Can a nonsingular matrix be closely approximated by a singular matrix?

**Exercise 18** (H-J 5.8.P10)**.** If the spectral norm is used, show that $\kappa(A^*A) = \kappa(AA^*) = \kappa(A^2)$ Explain why the problem of solving $A^*Ax = y$ may be intrinsically less tractable numerically than the problem of solving $Ax = z$.

**Exercise 19** (H-J 6.1.P6)**.** Recall the notation for the deleted absolute row sums of $A = [a_{ij}] \in M_n$:

$$R_i'(A) = \sum_{i \neq j} |a_{ij}|, \quad i = 1, \ldots, n.$$

Suppose $|a_{ii}| > R_i'$ for $k$ different values of $i$. Use properties of principal submatrices of $A$ to show that rank $A \geq k$.

**Exercise 20** (H-J 6.3.P4(a))**.** Let $A \in M_n$ be normal, let $\mathcal{S}$ be a given $k$-dimensional subspace of $\mathbb{C}^n$, and let $\gamma \in \mathbb{C}$ and $\delta > 0$ be given. If $||Ax - \gamma x||_2 \leq \delta$ for every unit vector $x \in \mathcal{S}$, show that there are at least $k$ eigenvalues of $A$ in the disc $\{z \in \mathbb{C} \mid |z - \gamma| \leq \delta\}$.

**Exercise 21** (H-J 7.3.P7: the *Moore–Penrose generalized inverse.*)**.** Let $A \in M_{m,n}$ and let $A = V\Sigma W^*$ be a singular value decomposition. Define $A^\dagger = W\Sigma^\dagger V^*$, in which $\Sigma^\dagger$ is obtained from $\Sigma$ by first replacing each nonzero singular value with its inverse and then transposing. Show that

(1) $AA^\dagger$ and $A^\dagger A$ are Hermitian.
(2) $AA^\dagger A = A$
(3) $A^\dagger AA^\dagger = A^\dagger$
(4) $A^\dagger = A^{-1}$ if $A$ is square and nonsingular.
(5) $(A^\dagger)^\dagger = A$.
(6) $A^\dagger$ is uniquely determined by properties (1-3).