**STAT 7600/7610 Mathematical Statistics Preliminary Exam**

**August 17, 2018**

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**NAME:** ________________________________

1. It is a closed-book in-class exam.
2. Calculator is allowed and will be provided by the proctor.
3. The proctor will provide as many blank sheets of paper as you need.
4. Show your work to recieve full credit. Highlight your final answer.
5. Solve any five problems out of eight. Only turn in the five problems you want to be graded or otherwise clearly indicate which five problems you are submitting for evaulation.
6. Turn in your the exam paper (the three typeset pages handed to you) along with your work-sheets stabled to the back. Please place the problems in numerical order and label each at the top.
7. Total points are 50 with 10 points for each problem.
8. Time: 240 minutes. (8:00am-12:00(noon), August 17, 2018).

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
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1. Suppose $X_1, X_2, \ldots, X_n$ are a random sample of size $n$ from an exponential $(\theta)$ population.

   (a) Show that the uniformly most powerful (UMP) test for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ can be expressed as

   $$\phi(\bar{X}) = I\left(\frac{2n\bar{X}}{\theta_0} > \chi^2_\alpha(2n)\right)$$

   (b) Derive the power function $\beta(\theta)$ and simplify. Use all appropriate notation.

   (c) Show that the following is a $1 - \alpha$ confidence interval using the *pivot method*

   $$\left[\frac{2n\bar{X}}{\chi^2_{\alpha/2}(2n)}, \frac{2n\bar{X}}{\chi^2_{1-\alpha/2}(2n)}\right]$$

2. Let $Y_1, \ldots, Y_n$ be a random sample from a distribution with probability mass function $f(y) = \rho(1-\rho)^y$, for $y = 0, 1, 2, \ldots$ and $0 < \rho < 1$. Then $E(Y) = (1-\rho)/\rho$ and $Var(Y) = (1-\rho)/\rho^2$. Find the method of moments estimator of $\rho$ and determine its asymptotic distribution.

3. Let $X$ be a single observation from a normal distribution with mean $\theta$ and variance $\theta^2$, where $\theta > 0$. Find the maximum likelihood estimator of $\theta^2$.

4. Suppose $X_1, \ldots, X_n$ are iid exponential$(\mu, \sigma)$, i.e., $(1/\sigma)e^{-(x-\mu)/\sigma}I(x \geq \mu)$, where $-\infty < \mu < \infty$ and $\sigma > 0$. We know that $\hat{\boldsymbol{\theta}} = T(\mathbf{X}) = (T_1(X), T_2(X)) = (X_{(1)}, \bar{X} - X_{(1)})$ is the *maximum likelihood estimator* (MLE) for $\boldsymbol{\theta} = (\mu, \sigma)$, where $\bar{X} = \sum X_i/n$ and $X_{(1)} = \min\{X_1, \ldots, X_n\}$. Also, $\hat{\boldsymbol{\theta}}$ is *complete* and *minimally sufficient* for $\boldsymbol{\theta}$.

   (a) Are $T_1$ and $T_2$ independent? Prove your answer.

   (b) Find the marginal distribution of $T_1$ and name this distributions in precise terms.

   (c) Find unbiased estimators of $\mu$ and $\sigma$, based on $T_1$ and $T_2$.

   (d) Are the estimators in (c) UMVUE? Explain/Show.

5. Let $X_1, \ldots, X_n$ be iid $n(\theta, \sigma^2)$, both $\theta$ and $\sigma$ unknown. We are interested in testing

   $$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.$$

   Recall, the unrestricted MLEs are $\hat{\theta} = \bar{X}$ and $\hat{\sigma}^2 = \sum(X_i - \bar{X})^2/n$. The restricted MLE's are $\hat{\theta}_0 = \theta_0$ and $\hat{\sigma}_0^2 = \sum(X_i - \theta_0)^2/n$.

   (a) Show that the test that rejects $H_0$ when $|\bar{X} - \theta_0| > t_{n-1,\alpha/2}\sqrt{S^2/n}$ is a test of size $\alpha$.

   (b) Show that the test in part (a) can be derived as a *likelihood ratio test* (LRT).

   (c) For $\sigma$ unknown and $\theta$ known, find a Wald statistic for testing $H_0 : \sigma = \sigma_0$.

6. Suppose that the random variables $Y_1, Y_2 \ldots, Y_n$ satisfy $Y_i = \beta x_i + \epsilon_i,\ i = 1, 2, \ldots, n$, where $x_1, x_2, \ldots, x_n$ are fixed constants (real numbers), $-\infty < \beta < \infty$, and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are iid $N(0, \sigma^2)$, $\sigma^2 > 0$ unknown.

   (a) Show that $(\sum Y_i^2, \sum x_i Y_i)$ is a sufficient statistic for $(\beta, \sigma^2)$.

   (b) Show that the MLE of $\beta$ is $\hat{\beta} = \dfrac{\sum x_i Y_i}{\sum x_i^2}$ and show that it is an unbiased estimator of $\beta$.

   (c) Derive the exact distribution of $\hat{\beta}$.

7. Suppose $X_1, X_2, \ldots, X_n$ are iid $N(\theta, 1)$. Let $\tau(\theta) = \theta^2$ . Recall that $\hat{\theta} = \bar{X}$ is MLE for $\theta$.

   (a) Compute the Cramer-Rao Lower Bound for any unbiased estimator of $\tau(\theta)$ from this family of distributions.

   (b) Find the MLE for $\tau(\theta) = \theta^2$. Note, compute the expected value of this MLE. Show your work/explanation.

   (c) Let $\hat{\tau} = \hat{\tau}(\theta) = \bar{X}^2 - \dfrac{1}{n}$. It can be shown (you don't have to show) that $\text{var}\left(\bar{X}^2\right) = \dfrac{2}{n^2} + \dfrac{4\theta^2}{n}$, derive the mean and variance of $\hat{\tau}$.

   (d) Find the UMVUE for $\theta^2$. Show work/explanation.

8. Suppose $X_1, X_2, \ldots, X_n$ are a random sample of size $n$ from a Beta $(\theta, 1)$, $\theta > 0$, population. Recall that the MOM and MLE estimators were shown to be

$$\hat{\theta}_{MOM} = \frac{\bar{X}}{1 - \bar{X}} \quad \text{and} \quad \hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln X_i}.$$

   (a) Show that $\hat{\theta}_{MOM}$ and $\hat{\theta}_{MLE}$ satisfy

$$\sqrt{n}(\hat{\theta}_{MOM} - \theta) \longrightarrow N\left(0, \frac{\theta(\theta+1)^2}{\theta+2}\right) \quad \text{and} \quad \sqrt{n}(\hat{\theta}_{MLE} - \theta) \longrightarrow N\left(0, \theta^2\right)$$

   (b) Calculate the asymptotic relative efficiency (ARE) and comment. Which one is more efficient? Explain.