MATH 7820-7830 Applied Stochastic Processes
Prelim Exam, August 2022

To pass you need to get at least 65%

Name:
PID:

1. (20pts)

(a) (10pts) Let X be a random variable and $c > 0$. Prove that

$$P(X > c) \leq M_X(t)e^{-ct}$$

where $M_X(\cdot)$ is the moment generating function of X.

(b) (10pts) Using the result in part **(a)** try to give the best possible estimate you can get for $P(X > 200)$ where X is a Poisson variable with parameter 50.

2. (10pts) Let X_n be a Negative Binomial random variable with parameters r and $p = n/\lambda$. Find a random variable X so that $\frac{1}{n}X_n \implies X$ (converges in distribution). Justify your answer.

3. (10pts) Let $N(t)$ be a Poisson process with rate λ, with arrival times $\{S_n, n = 0, 1, \cdots\}$. Evaluate the expected sum of squares of the arrival times occurring before t,

$$E(t) = E\Big(\sum_{n=1}^{N(t)} S_n^2\Big)$$

where we define $\sum_{n=1}^{0} S_n = 0$

4. (25pts) Let $N(t)$ be a renewal process with cycle times T_n with distribution F and renewal function $m(t) = E(N(t))$.

(a) Show that renewal function $m(t) = E(N(t))$ satisfies the renewal equation. In other words, show that $m(t)$ satisfies

$$m(t) = F(t) + \int_0^t m(t-x)dF(x).$$

(b) Define what it means for a random process $X(t)$ to be T_1-shift invariant.

(c) What is the renewal equation for $E(X(t))$ if $X(t)$ is T_1-shift invariant? What is the solution of this renewal equation?

(d) Show that the event $A(t) = \{t - S_{N(t)} \leq y\}$ is T_1-shift invariant for any $y \geq 0$.

(e) What is the distribution of $A(t)$ using parts **(c)** and **(d)**? What is the limiting distribution of $A(t)$?

5. (10pts) The life of a car is a random variable with distribution F. An individual has a policy of trading in his car either when it fails or reaches the age A. Let $R(A)$ denote the resale value of an A-year-old car. There is no resale value of a failed car. Let C_1 denote the cost of a new car and suppose that an additional cost C_2 is incurred whenever the car fails.

(a) Say that a cycle begins each time a new car is purchased. Compute the long-run average cost per unit time.

(b) Say that a cycle begins each time a car in use fails. Compute the long-run average cost per unit time.

6. (40pts) Consider the birth or death model with transition probabilities $p(i, i+1) = p_i$ for $i = 0, 1, 2, \cdots$, $p(i, 0) = q_i$ for $i = 0, 1, 2, \cdots$, here $0 < p_i, q_i < 1$ and $p_i + q_i = 1$.

(a) (5 pts) Find the the transition matrix, and give the diagram of transitions.

(b) (10pts) Find the stationary distribution in terms of one of the coordinates.

(c) (5pts) Show that the stationary distribution exists if and only if

$$C = 1 + p_0 + p_0 p_1 + p_0 p_1 p_2 + \cdots < \infty.$$

(d) (5pts) What is the stationary distribution?

(e) (5pts) When is the chain positive recurrent?

(f) (5pts) If $p_i = p$ is a constant what is the stationary distribution?

(g) (5pts) Suppose the chain starts at 0. What is the expected first return time to state 0 if $p_i = p$ a constant?

7. (10pts) Demands, each is of 1 or 2 units with equal probability, arrive at a store according to a Poisson process of rate 1 and are fulfilled immediately when there is enough inventory. When the inventory level falls to 0, it is replenished to 3 units, but it takes an exponential time of mean 1 to complete the replenishment, during which time, demands are lost. Note that if 2 units of demand arrive when the inventory level is 1, then 1 unit of demand is lost.

(a) Find the long run fraction of time when the inventory level is at 0.

(b) What fraction of demand units are lost in the long run?

8. (30pts) Let $X(t)$ be a continuous time Markov chain.

(a) State the Chapman-Kolmogorov identity.

(b) Prove the Chapman-Kolmogorov identity.

(c) Let q_j be the rate out of state j and q_{ij} be the rate from i to j. Show that the $\{\pi_j\}$ satisfy the balance equation

$$\pi_j q_j = \sum_{i \neq j} \pi_i q_{ij} \quad \text{together with} \quad \sum_j \pi_j = 1$$

if and only if $\{\pi_j\}$ satisfy

$$\pi_j = \sum_i \pi_i P_{ij}(t) \quad \text{for} \quad \text{all} t \geq 0.$$

9. (10pts) Let $B(t)$ be a standard Brownian motion and define $W(t) = B(a^2 t)/a$ for $a > 0$. Verify that $W(t)$ is also Brownian motion.

10. (15pts) Let $B(t)$ be a standard Brownian motion. The event that $B(t)$ has a zero crossing between s and t is $A(s,t) = \{B(u) = 0 \, \text{for some} \; u \; \text{with} \; s < u < t\}$.

(a) Let $\tau_x = \inf\{u > 0 : B(u) = x\}$ for $x > 0$. Find $P(\tau_x \leq t)$ in terms of the standard normal distribution function $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du$.

(b) By conditioning on $B(s)$, find an expression for $P(A(s,t))$. Evaluate this expression using the identity

$$\int_0^\infty e^{-v^2/2s} \left\{ \int_v^\infty e^{-u^2/2(t-s)} du \right\} dv = \sqrt{s(t-s)} \arccos \sqrt{s/t}$$

where $\arccos$ is the inverse of the cosine function. You are not asked to prove this identity, but you are asked to use it.