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Effect of the Diamagnetism on the Number of Observable Hydrogen Lines in Plasmas

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ABSTRACT: For relatively strong magnetic fields, hydrogen atoms can have delocalized bound states of almost macroscopic dimensions. We study the influence of the delocalized states of hydrogen atoms on the number of observable hydrogen lines in strongly magnetized plasmas. We show that for sufficiently large values of the pseudomomentum K (K being the integral of the motion controlling the separation of the center-of-mass and the relative motions), this effect dominates other factors potentially influencing the number of observable hydrogen lines in strongly magnetized plasmas. We provide examples for the conditions of edge plasmas of contemporary and future tokamaks, as well as for DA white dwarfs. We also demonstrate that our results open up an avenue for the experimental determination of the pseudomomentum K. This is the first proposed method for the experimental determination of the pseudomomentum – to the best of our knowledge.

***Keywords*:** strong magnetic fields; center-of-mass effects; diamagnetic ter; hydrogen spectral lines; diagnostic of pseudomomentum

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1. INTRODUCTION

There is plenty of studies showing that for hydrogenic atoms/ions in a uniform magnetic field **B**, the center-of-mass motion and the relative (internal) motion are coupled by the magnetic field and, rigorously speaking, cannot be separated – see, e.g., papers [1-3] and references therein. A pseudoseparation is possible for hydrogen atoms. It leads to a Hamiltonian for the relative motion that depends on a center-of-mass integral of the motion **K** called pseudomomentum, but does not depend on the center of mass coordinate [3]. We remind that the pseudomomentum **K** is the canonical variable conjugated to the center of mass coordinate.

In paper [4] it was shown that in a non-uniform electric field, the center-of-mass motion and the relative motion cannot be separated exactly, but only by using the approximate analytical method of separating rapid and slow subsystems. It was further demonstrated in paper [4] that the outcome is significant changes in the Stark broadening of hydrogen lines in plasmas, the effects having practical applications to magnetic fusion, radiofrequency discharges, and flare stars.

Under a uniform magnetic field, the strongest center-of-mass effect is that the diamagnetic potential term in the Hamiltonian for the relative motion can become responsible for the formation of an additional potential well (hereafter, B-well) – far away from the hydrogen nucleus (proton). For relatively strong magnetic fields B, the new bound states inside the B-well are delocalized states of almost macroscopic dimensions.

In the present paper we consider the effect of the B-well on the number of observable hydrogen lines in strongly magnetized plasmas. We show that for sufficiently large values of the pseudomomentum K, this effect dominates other factors potentially influencing the number of observable hydrogen lines. We provide examples for the conditions of edge plasmas of contemporary and future tokamaks, as well as for DA white dwarfs.

2. DETAILS OF THE EFFECT

In the Hamiltonian of the relative (internal) motion for the hydrogen atom in a magnetic field **B**, the potential energy has the form (see, e.g., Schmelcher-Cederbaum paper [5], Eq. (6))

$$V = [e^2/(2Mc^2)](\mathbf{B}\mathrm{x}\mathbf{r})^2 - [|e|/(Mc)\mathbf{B}\mathrm{x}\mathbf{K})\mathbf{r} - e^2/r, \qquad (1)$$

where **K** is the pseudomomentum, M is the mass of the hydrogen atom, c is the speed of light, and e is the electron charge. We follow paper [5] in choosing $e < 0$; we also note that in paper [5] it was set $c = 1$. In Eq. (1), $(\mathbf{B}\mathrm{x}\mathbf{K})\mathbf{r}$ stands for the scalar product (also known as the dot-product) of vector **r** and vector (**B**x**K**). The first term in Eq. (1) is the diamagnetic one; the second term is due to the motional Stark effect; the third term represents the Coulomb interaction.

We also assume the same configuration as chosen in paper [5]: **B** = (0, 0, B), **K** = (0, K, 0), where K > 0. Then Eq. (1) takes the form:

$$V = [e^2B^2/(2Mc^2)](x^2 + y^2) + [|e|BK/(Mc)]x - e^2/(x^2 + y^2 + z^2)^{1/2}. \qquad (2)$$

For simplifying equations, we introduce the following scaled potential energy V_s:

$$V_s = Mc^2V/(e^2B^2). \qquad (3)$$

Then Eq. (2) can be rewritten as follows:

$$V_s = (x^2 + y^2)/2 + [Kc/(|e|B)]x - Mc^2/[B^2(x^2 + y^2 + z^2)^{1/2}]. \qquad (4)$$

By equating partial derivatives of V_s with respect to y and z to zeros, we find that in the y- and z-coordinates the minimum of the potential occurs at $y = z = 0$ (as in paper [5]). The partial derivative of V_s with respect to x that we calculate for finding extrema of the potential with respect to the x-coordinate, has the form:

$$\partial V_s/\partial x = x + Kc/(|e|B) + Mc^2x/[B^2(x^2 + y^2 + z^2)^{3/2}]. \qquad (5)$$

At $y = z = 0$, Eq. (5) becomes:

$$\partial V_s/\partial x = x + Kc/(|e|B) + (Mc^2/B^2)(\text{sign } x)/x^2. \qquad (6)$$

We remind that sign x = 1 for x > 0 or sign x = –1 for x < 0.

On equating $\partial V_s/\partial x$ from Eq. (6) to zero, we arrive to the following equation:

$$x^3 + [Kc/(|e|B)]x^2 + (Mc^2/B^2)(\text{sign } x) = 0. \qquad (7)$$

Thus, for x > 0 and for x < 0, Eq. (7) leads to two different equations.

For x > 0, Eq. (7) becomes:

$$x^3 + [Kc/(|e|B)]x^2 + (Mc^2/B^2) = 0. \qquad (8)$$

Obviously, Eq. (8) does not have positive roots.

For x < 0, Eq. (7) leads to

$$|x|^3 - [Kc/(|e|B)]|x|^2 + (Mc^2/B^2) = 0. \qquad (9)$$

Equation (9) is equivalent to Eq. (8b) from paper [5]; we remind that in paper [5] it was set c = 1 and e = –1.

The polynomial in Eq. (9) has either two or zero real roots. Thus, the total number of roots of Eq. (7) is also either two or zero – since there are no positive roots.

We note in passing that the authors of paper [5] erroneously stated that $\partial V/\partial x$, calculated at y = z = 0, can have three real roots. Their error originates from the fact that they missed the factor (sign x) in the corresponding equation.

We introduce the scaled magnetic field b and the scaled pseudomomentum k, as follows

$$b = B/(cM^{1/2}), \qquad k = K/(|e|M^{1/2}), \tag{10}$$

where b has the dimension of $cm^{-3/2}$ and k has the dimension of $cm^{-1/2}$. Below, while using particular numerical values of b and k, we omit the dimensions for brevity.

With these notations, Eq. (9) simplifies as follows:

$$|x|^3 - (k/b)|x|^2 + 1/b^2 = 0. \tag{11}$$

The discriminant Δ of this cubic equation is

$$\Delta = (4k^3 - 27b)/b^5. \tag{12}$$

So, Eq. (11) has two distinct real negative roots if $\Delta > 0$, i.e. if

$$k > k_{min}, \qquad k_{min} = 3(b/4)^{1/3} \approx 1.89\, b^{1/3}. \tag{13}$$

In the non-scaled units:

$$K_{min}(a.u.) \approx 0.375\, [B(Tesla)]^{1/3}. \tag{14}$$

We remind that the atomic unit, in which K_{min} is measured in the left side of Eq, (14), is the atomic unit of any linear momentum – it is equal to

$$m_e|e|^2/\hbar \approx 1.99 \times 10^{-19} \text{ g cm/s}. \tag{15}$$

The exact analytical results for the two real roots x_1 and x_2 of Eq. (11) are as follows:

$$x_1 = -k/(3b) - (1 + 3^{1/2}i)\, k^2/\{2^{2/3}3b[27b - 2k^3 + 3^{3/2}(27b^2 - 4k^3b)^{1/2}]^{1/3}\} - (1 - 3^{1/2}i)\,[27b - 2k^3 + 3^{3/2}(27b^2 - 4k^3b)^{1/2}]^{1/3}/(2^{1/3}6b). \tag{16}$$

$$x_2 = -k/(3b) - (1 - 3^{1/2}i)\, k^2/\{2^{2/3}3b[27b - 2k^3 + 3^{3/2}(27b^2 - 4k^3b)^{1/2}]^{1/3}\} - (1 + 3^{1/2}i)\,[27b - 2k^3 + 3^{3/2}(27b^2 - 4k^3b)^{1/2}]^{1/3}/(2^{1/3}6b). \tag{17}$$

It should be emphasized that, despite the presence of the imaginary unit i in Eqs. (16) and (17), they yield real numbers for x_1 and x_2 under the condition (13).

For sufficiently large values of the pseudomomentum K, such that

$$k >> b^{1/3}, \tag{18}$$

the expressions for the roots of Eq. (11) simplify to:

$$x_1 \approx -k/b, \quad x_2 \approx -1/(kb)^{1/2} \tag{19}$$

It is easy to find out that at $x = x_1$ one has $\partial^2 V_s/\partial x^2 > 0$, so that the scaled potential energy V_s has a minimum at $x = x_1$ (under condition (13)). At $x = x_2$ one has $\partial^2 V_s/\partial x^2 < 0$, so that the scaled potential energy V_s has a maximum at $x = x_2$ (under condition (13)).

The scaled potential energy V_s at y = z = 0 has the form:

$$V_s = x^2/2 + kx/b - 1/(b^2|x|). \tag{20}$$

Figure 1 shows the plot of the scaled potential energy V_s from Eq. (20) versus the coordinate x (in cm) at the scaled magnetic field b = 1.3x10^6 (corresponding to B = 5 Tesla) for the scaled pseudomomentum: k = 960 corresponding to K = 3 a.u. (solid line). The dashed line corresponds to the energy E_{top} at the top of the potential barrier.

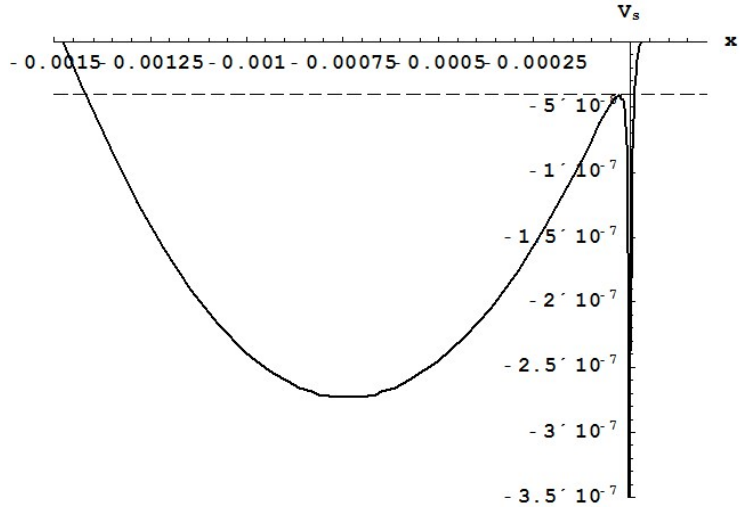

Fig. 1. The plot of the scaled potential energy V_s from Eq. (20) versus the coordinate x (in cm) at the scaled magnetic field b = 1.3x10^6 (corresponding to B = 5 Tesla) for the scaled pseudomomentum: k = 960 corresponding to K = 3 a.u. (solid line). The dashed line corresponds to the energy at the top of the potential barrier.

Here we come to the central point. For hydrogen energy levels below the top of the potential barrier ($E < E_{top}$), the atomic electron is confined in a relatively narrow potential well. However, for the energy levels at or above the top of the potential barrier, the width of the potential well increases by several orders of magnitude (for sufficiently large values of the pseudomomentum K). According to the uncertainty relation, this means that in the latter case, the spacing of the energy levels decreases by several orders of magnitude. Since these energy levels have a finite width (e.g., due to the Stark broadening in plasmas and the natural broadening), the dramatic decrease of the spacing between these energy levels creates a quasi-continuum out of them and thus makes the atomic electron to be practically free from the proton. This is equivalent to the ionization on the atom. Therefore, this mechanism limits the number of the discrete energy levels E_n by the condition:

$$E_{top} > E_n = m_e e^4/(2\,\hbar^2 n^2). \tag{21}$$

At the top of the potential barrier (i.e., at $x = x_2$), under the condition (18) the scaled potential energy is

$$V_{s,top} \approx -2k^{1/2}/b^{3/2}. \tag{22}$$

The corresponding value of the non-scaled potential energy is

$$E_{top} = V_{top} \approx -2|e|^{3/2}[KB/(Mc)]^{1/2}. \tag{23}$$

From formulas (21) and (23), it is easy to obtain the following expression for the maximum principal quantum number $n_{max,B}$ (caused by the presence of the B-well):

$$n_{max,B} = [|e|Mc/(16a_0^2BK)]^{1/4}, \tag{24}$$

where a_0 is the Bohr radius. The practical formula for $n_{max,B}$ is:

$$n_{max,B} \approx 72/(BK)^{1/4}, \tag{25}$$

where B is in Tesla and K is in a.u.

In plasmas, the average Lorentz field $E_{LT} = Bv_T/c$ (where $v_T = (2T/M)^{1/2}$ is the atomic thermal velocity) can exceed the most probable ion microfield E_i when the magnetic field B exceeds the following critical value [6, 7]:

$$B_c = 4.35x10^{-9}N_e^{2/3}/T^{1/2}, \tag{26}$$

where B is in Tesla, the electron density N_e is in cm^{-3}, and the temperature T is in eV. The condition $B > B_c$ is fulfilled, e.g., for the edge plasmas of tokamaks and in the atmospheres of white dwarfs. In papers [6, 7] it was shown that under the condition $B > B_c$, the principal quantum number $n_{max,L}$ of the last observable hydrogen line (while disregarding the effect of the B-well) is:

$$n_{max, L} = [m_e^2|e|^5c/(3\hbar^4v_TB)]^{1/5} \tag{27}$$

(the letter L in the subscript stands for Lorentz field).

By comparing Eqs. (24) and (27), it is easy to find out that the presence of the B-well controls the number of the observable hydrogen lines (i.e., $n_{max,B} < n_{max,L}$) for sufficiently large values of the pseudomomentum

$$K > K_{crit} = (M/16)[81|e|cv_T^4/(a_0^2B)]^{1/5}. \tag{28}$$

The practical formula for the critical pseudomomentum value is:

$$K_{crit} \approx 57(T^2/B)^{1/5}, \tag{29}$$

where K_{crit} is in a.u., T is in eV, and B is in Tesla.

Now let us consider some examples. For B = 5 Tesla (typical for the conditions of the contemporary tokamaks), Eq. (25) yields $n_{max,B} = 15$ for K = 100 a.u. or $n_{max,B} = 11$ for K = 300 a.u.

Future tokamaks could use superconductive magnets, such as, e.g., the MIT fusion magnet creating B = 20 Tesla [8]. For B = 20 Tesla, Eq. (25) yields $n_{max,B} = 10$ for K = 100 a.u. or $n_{max,B} = 8$ for K = 300 a.u.

The atmospheres of DA white dwarfs (i.e., of the white dwarfs emitting hydrogen lines) are characterized by magnetic fields B from 100 to 100000 Tesla. For B = 2000 Tesla, Eq. (25) yields $n_{max,B} = 4$ for K = 50 a.u.

In all of the above examples, the principal quantum number of the last observable hydrogen line is controlled by $n_{max,B} < n_{max,L}$.

Thus, the primary effect of the diamagnetic term in the Hamiltonian is the creation of the B-well causing the decrease of the number of observable hydrogen lines. Compared to this primary effect, other effects of the diamagnetic term – those discussed in paper [9] – are just minor, secondary outcome.

Our results open up an avenue for the experimental determination of the pseudomomentum K. Indeed, from Eq. (25) one gets:

$$K(a.u.) \approx (n_{max, B}/72)^4/B(Tesla). \tag{30}$$

Thus, from the experimental values of the magnetic field B and the number $n_{max,B}$ of the last observable hydrogen line, it is possible to deduce the value of the pseudomomentum by using Eq. (30). This is the first proposed method for the experimental determination of the psudomomentum – to the best of our knowledge.

3. CONCLUSIONS

We analyzed the influence of the delocalized states of hydrogen atoms (the B-well) on the number of observable hydrogen lines in strongly magnetized plasmas. We demonstrated that for sufficiently large values of the pseudomomentum K (K being the integral of the motion controlling the separation of the center-of-mass and the relative motions), this effect dominates other factors potentially affecting the number of observable hydrogen lines. We gave examples for the conditions of edge plasmas of contemporary and future tokamaks, as well as for DA white dwarfs.

We showed that our results open up an avenue for the experimental determination of the pseudomomentum K. This is the first proposed method for the experimental determination of the psudomomentum – to the best of our knowledge.

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