IRAMP
14(2), July 2023, pp. 69-79

ISSN: 2229-39-44

Generalization of Richardson's t^3 scaling law for hydrodynamics of fluids and gases, based on Biberman-Holstein approach to radiative transfer

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ABSTRACT: A scaling that generalizes the empiric Richardson's t^3 scaling law for the mean square of the mutual separation of a pair of particles in a fluid or gaseous medium is generalized to the case of longer times but still before the onset of the diffusion regime. This corresponds to the description of combined regime of Lévy flights and Lévy walks for hydrodynamic turbulence in fluids and gases. The general concept of Lévy flights proposed by Shlesinger and colleagues for turbulence was formulated in the frame of the nonlocal transport which is based on the Biberman-Holstein approach to the transfer of excitation of a medium by photons, generalized to take into account the finiteness of the velocity of excitation carriers. The application of this approach to describing the nonlocality of hydrodynamics of fluids or gases is preceded by the success of its application to observations of plasma density fluctuations moving across a strong magnetic field in tokamaks.

1. INTRODUCTION

The nonlocality (superdiffusion) of turbulence is expressed in the empirical Richardson t^3 scaling law [1] for the turbulent relative dispersion, i.e. for the mean square of the mutual separation of a pair of particles, $r_{pair}(t)$, in a fluid or gaseous medium,

$$r_{pair}^2(t) \propto t^3. \quad (1)$$

This scaling was obtained within the framework of the diffusion model proposed by Richardson [1] (with the diffusion coefficient K depending on distance r, $K \propto r^{4/3}$), suggested by experimental data for atmospheric turbulence. Subsequently, the diffusion approach was developed in the direction of complicating the dependence of the diffusion coefficient. Along with this, the awareness of the phenomena of superdiffusion of turbulent dispersion came onto the scene, although Bachelor's scaling [2], $r_{pair}^2(t) \propto t^2$, for ballistic motions is discussed only in connection with the initial stage of the separation of a pair of test particles. As a result, Richardson's empirical scaling for pair correlations was derived in various models (see, for example, review [3]), but at present, a nonlocal approach based on superdiffusion models is considered more adequate, in which, by the definition of superdiffusion, $r_{pair}^2(t) \propto t^\beta$, where $\beta > 1$. Within this framework, the concept of turbulence nonlocality is further developed, including attempts to reassess the Richardson scaling (1) itself (see [3]).

The key step in the development of the theory of nonlocality of various processes in physics and other sciences based on the concept of "Lévy flights", introduced in [4] by Mandelbrot [5] (see [4, 6–8]), and Levy walks [9–11], which generalize Lévy flights to the case of taking into account the finite velocity of carriers, was the idea of Shlesinger and colleagues [10] on the possibility of describing the nonlocality of turbulence using a linear integro-differential equation with a kernel, slowly decreasing with distance. This approach suggests that the essentially nonlinear dynamics of what we call turbulence can be reduced to the evolution of a statistical ensemble of carriers that have a large free path and, in fluid mechanics, can be associated with such stable objects as vortices/eddies. With this approach, the severity of complex nonlinear dynamics is transferred to the axiom of the existence of long-lived long-range motions in the medium, which are carriers of perturbations of this medium relative to some stable macroscopic state of the medium. The nonlocal transport of perturbations of the medium is assumed to be described by linear kinetic equations with the kernel of the integral operator, slowly decreasing with distance and belonging to the class of Lévy distributions.

In [12], an approach similar in spirit to [10] was proposed. The formalism of the type of the Biberman-Holstein equation [13, 14] for the transfer of excitation by photons in gases and plasmas (for details, see, for example, [15, 16]), generalized to take into account the finiteness of the velocity of excitation carriers, was taken as a basis. Before formulating the model [12], it is appropriate to briefly describe the ideology of the Biberman-Holstein approach.

The basic equation for the excitation density of a medium in the problem of resonant radiation transfer is formulated for the density of excited atoms or ions $f(\boldsymbol{r}, t)$. The Biberman-Holstein approach uses the approximation of complete redistribution (CRD) in photon's frequency in the elementary act of absorption of a photon by an atom or ion and re-emission of a photon in the same spectral line for the model of a two-level atom or ion, which is acceptable for the transfer of resonant radiation (generalization to the case of interdependent transfer of radiation in many spectral lines is easily feasible). The kinetic equation for the excitation density of the medium, in the case of stationary motionless unexcited medium, has the form:

$$\frac{\partial f(\boldsymbol{r},t)}{\partial t} = \frac{1}{\tau}\int_V W(|\boldsymbol{r}-\boldsymbol{r}_1|) f(\boldsymbol{r}_1,t)\,dV_1 - \left(\frac{1}{\tau}+\sigma\right) f(\boldsymbol{r},t) + q(\boldsymbol{r},t), \tag{2}$$

where σ is the lifetime of an excited atomic state with respect to spontaneous radiative decay; σ is the rate of collisional excitation quenching; q is the source of excitation of atoms, which is different from the excitation due to the absorption of a resonant photon (i.e., the source of collisional excitation). The kernel W is determined by the (normalized) spectral distribution of line emission source, ε_ν, and the absorption coefficient k_ν, which is the inverse free path length (for the theory of spectral line shapes see [17–23]). Here ε_ν is specified as a function depending only on the parameters of the emitted photon (frequency and direction) and independent of the parameters of the absorbed photon, if the excited state was formed as a result of the absorption of the photon (i.e., there is the loss of memory by the excited atom about the prehistory of excitation). This feature of is a consequence of the CRD approximation mentioned above. In a homogeneous medium, W depends on the distance between the points of emission and absorption of a photon:

$$W(\rho) = -\frac{1}{4\pi\rho^2}\frac{dT(\rho)}{d\rho} \equiv \frac{1}{4\pi\rho^2} W_{step}(\rho), \quad T(\rho) = \int_0^\infty \varepsilon_\nu \, exp(-k_\nu \rho)\, d\nu. \tag{3}$$

where the function $T(\rho)$, called the Holstein function (see, for example, [16]), is the probability that a photon will freely travel a distance not less than ñ without absorption. Thus, this function defines the distribution function for the carrier free path length W_{step} (step-length PDF). Accordingly, the kernel W, Eq. (3), of the integral transport equation (2) in three-dimensional coordinate space specifies the probability that the emitted photon will be absorbed at a distance ñ from the point of photon's birth. For practically interesting spectral line broadening mechanisms, including the Doppler effect and various mechanisms that give the Lorentzian form of the spectral line shape (spontaneous radiative decay, collisional perturbation of the excited state, including the dynamic Stark effect), the Holstein function at distances corresponding to large optical thicknesses, defined by the value of the absorption coefficient at the center of the spectral line k_0, has a slow, power-law decay. For the Lorentzian (4) and Gaussian (i.e. Maxwellian Doppler) (5) shapes of the spectral line, one has [14]:

$$T(\rho) \sim \frac{1}{\sqrt{\pi\kappa_0\rho}}, \tag{4}$$

$$T(\rho) \sim \frac{1}{\kappa_0 \rho \sqrt{\pi\, ln(\kappa_0 \rho)}}, \qquad \kappa_0 \rho \gg 1. \tag{5}$$

Nonlocality (superdiffusion) of radiation transfer, described by the Biberman-Holstein equation (2), requires a special definition of the average time $\bar{t}(r)$, required for a photon to pass the distance r from an instantaneous point source $q(\boldsymbol{r}, t) = \delta(\boldsymbol{r} - \boldsymbol{r}_0)\delta(t - t_0)$. The commonly used concept of the average distance traveled by a photon in a given time turns out to be inapplicable in the case of superdiffusion, because the function $f(r, t)$ falls too slowly with increasing distance from the primary source, and therefore the integral that determines the mean square of the displacement $\overline{r^2}$ diverges. The definition of $\bar{t}(r)$ corresponding to the case of superdiffusion was given in [24] and has the following form:

$$\bar{t} = \int_0^\infty dt \int_0^r dr_1 4\pi r_1^2 f(r_1, t). \tag{6}$$

In [24], analytical expressions for the asymptotic behavior of $\bar{t}(r)$ (6) were obtained for two forms of the spectral line shape. For the dispersion (Lorentzian) form of the spectral line shape, we have the motion of the effective excitation front of the medium corresponding to the acceleration ($r \propto t^2$),

$$\bar{t} = 3\tau\sqrt{\kappa_0 r/\pi} = 1.7\tau\sqrt{\kappa_0 r}, \quad t \gg \tau, \quad \kappa_0 r \gg 1, \tag{7}$$

and for the Doppler form of the spectral line shape, the corresponding motion is almost free ($r \propto t \ln(t/\tau)$):

$$\bar{t} = 1.4\tau\kappa_0 r\sqrt{ln(\kappa_0 r)}, \quad t \gg \tau, \quad \kappa_0 r \gg 1. \tag{8}$$

Subsequently, in [25], it was indicated that both cases (7) and (8) are covered by a single formula,

$$\bar{t}(\rho) \approx 1/T_{as}(\rho), \tag{9}$$

where $T_{as}(\rho)$ is given by (4) and (5).

When analyzing the transport for various mechanisms of spectral line broadening, it was shown [26] that the Biberman approximation [27], which makes it possible to reduce the equation integral with respect to coordinates to an algebraic one, is applicable the better, the slower the Holstein function decreases with increasing distance. The use of the approximation [27] in the theory of radiative transfer in one or many spectral lines is called the τ_{eff} method [28] or the Escape Probability method [29, 30].

Equation (2) is integral over spatial coordinates and is not reducible to a differential equation: the term "diffusion" in the titles of some articles, including [24], is explained only as a tribute to the then existing terminology and does not correspond to the mathematical apparatus of diffusion. When the sought-for function $f(r_1, t)$ is expanded under the space integral in a Taylor series, in the case of an infinite volume of the medium, the diffusion coefficient turns out to be infinite, and in the case of a finite volume, it depends on the size of the medium (see, for example, [16]), which in principle contradicts the concept of diffusion. Despite the fact that the term Lévy flights had not yet penetrated into the theory of radiative transfer at the time of publications [15, 16, 29, 30], the main transfer mechanism investigated in these theoretical approaches, in fact, refers specifically to Lévy flights.

An alternative to the Biberman-Holstein equation, which is widely used to describe laboratory plasma, is an approach often used in astrophysics, in which the pair of differential kinetic equations for photons and excited atoms/ions is reduced to an integral, in spatial variables, equation for the radiation intensity (see, e.g., [31–35]). Here, the dominant role of long free travels (i.e. Lévy flights) is also recognized, although usually without a proper name. The role of Lévy flights for light in the traditional radiative transfer in spectral lines was studied, for example, in [36]. Here, multiple scattering of near-resonant light in hot atomic vapors, characterized by Doppler broadening of the spectral line, experimentally confirmed theoretical studies [37], where it was shown that photon trajectories in the Biberman-Holstein model for the transfer of resonant radiation in spectral lines with Doppler, Lorentz, and Voigt broadening mechanisms contain Lévy flights.

The current state of the concept of Lévy flights and walks can be found in an extensive review [11], covering various fields of science. The latest results for the Green's function of the problem of nonstationary transfer of resonant radiation in the Biberman-Holstein model are presented in [38], including a generalization of the interpolated self-similarity method [39] to the case of a finite speed of light or other medium excitation carriers in [40–42]. Further progress in this direction led in [43] to the derivation of a unified approximate analytical description of the front of the nonstationary Green's function for transfers in the Lévy flights and Lévy walks modes.

The results of these works served as an impetus to the application of the Biberman-Holstein model in two directions. In Section 2, we briefly discuss the application to the interpretation of experiments on diagnosing plasma density fluctuations moving across a strong magnetic field in a tokamak [12] (as it turned out, tokamak plasma is a really turbulent medium in agreement with Richardson's t^3 law for hydrodynamic turbulence). In Section 3, a scaling is proposed, which generalizes Richardson's law to the combined mode of Lévy flights and Lévy walks for hydrodynamic turbulence. One of the arguments in favor of the Biberman-Holstein approach is that it is at least another way to derive Richardson's t^3 law (1).

2. SUPERDIFFUSION IN PLASMA TURBULENCE

The study of superdiffusion processes in plasmas is of great practical interest. Thus, in studies of magnetically confined plasma for controlled thermonuclear fusion (primarily in the most successful direction, namely, in tokamaks), it has long been recognized that the process of heat transfer across a strong magnetic field is anomalous in the sense that the heat diffusion coefficients reconstructed from experimental data by solving inverse problems in the framework of diffusion models of transfer significantly (by one and a half to two orders of magnitude) exceed the predictions of theories from first principles. Therefore, in the interpretation of experiments and predictive modeling of plasma behavior, phenomenological models suggested by the results of experiments are used. These models are, as a rule, diffusion models; based on differential equations of the Fokker-Planck type in space variables. Attempts to construct models of nonlocal transfer from first principles have not yet given the desired agreement with experiment. Let us point out, for example, the theory of non-stationary non-local heat transfer by longitudinal waves in plasma [44], brought to the final result for electron Bernstein waves (this method generalized the model of non-local transfer by electron and ion Bernstein waves [45] to the case of non-stationarity). Significantly greater success was achieved in the application of the above-mentioned Escape Probability method to the calculation of a separate component of the energy balance in tokamaks, namely, energy losses caused by the electron cyclotron radiation [46, 47]. This approach extended and modified the approach [48–50] to the heat transfer by electron cyclotron waves in thermonuclear plasmas (the current state of this issue can be found in [51]).

Let us consider a model of the microscopic dynamics of localized excitations of a medium (e.g., density perturbations) in a macroscopically quasi-homogeneous quasi-stationary medium. This model is an application of the Lévy walk concept to this class of problems. Similarly to the Biberman-Holstein model, we will consider two types of excitation of the medium, which can be transformed into each other. These two types can also be considered as different states (rest and motion) of the same object. This approach is applicable, for example, to biological migration, for which the applicability of the Lévy walk concept is well known (see, for example, review [11], section VI). In [61], a method was proposed and tested on synthetic experimental data for obtaining the Green's function of two-dimensional biological migration based on kinetic equations similar to those described below. The dominance of long-range movements (i.e., Lévy flights) corresponds to the fact that, for example, in search of food, animals try to escape from the place of the last stop as far as possible. Therefore, the trajectory of such motion topologically coincides with the trajectory of the excitation of the medium in the Biberman-Holstein model (compare, for example, Fig. 1 in [43], Fig. 1 in [52] and Fig. 4 in [6]). This motion was called "Lévy walk with rests" (Figure 1 in [11]). In the model we are considering, it is precisely such motions that are meant, but the common name is shortened to "Lévy walks".

Let us consider a system of nonlocal transport equations for the intensity $I_{\widetilde{\omega}}(\boldsymbol{r},\boldsymbol{n},t)$ (i.e., the energy flux density) of medium's excitation carriers (e.g., running fluctuations of density) and standing excitations of the medium (e.g., standing fluctuations of density) $f(\boldsymbol{r}, t)$:

$$\frac{\partial f(\boldsymbol{r},t)}{\partial t} = -\left(\frac{1}{\tau}+\sigma\right) f(\boldsymbol{r},t) + \int d\widetilde{\omega} \frac{\kappa_{\widetilde{\omega}}(v)}{\hbar\widetilde{\omega}} \int d\Omega(\boldsymbol{n})\, I_{\widetilde{\omega}}(\boldsymbol{r},\boldsymbol{v},t) + q(\boldsymbol{r},t), \tag{10}$$

$$\frac{1}{v}\frac{\partial I_{\widetilde{\omega}}(\boldsymbol{r},\boldsymbol{v},t)}{\partial t} + (\boldsymbol{n},\nabla) I_{\widetilde{\omega}}(\boldsymbol{r},\boldsymbol{v},t) = -\kappa_{\widetilde{\omega}}(v)\, I_{\widetilde{\omega}}(\boldsymbol{r},\boldsymbol{v},t) + \frac{\hbar\widetilde{\omega}}{\tau}\frac{P_{\widetilde{\omega}}(v)}{4\pi} f(\boldsymbol{r},t). \tag{11}$$

Let us describe the phenomenological functions and parameters included here, which should be restored by comparing the predictions of this model with experimental data.

Here $\boldsymbol{v}$ is the velocity of carriers of excitation, $\boldsymbol{v} = \boldsymbol{n}v$, which is assumed to be constant all the way between the point of birth and disappearance due to the transformation into a standing excitation of the medium. This means that we assume the existence of localized long-lived motions in a medium with a long free path, which are usually called solitons. An example of such motion in the two-dimensional case in plasmas and other media is the stable solutions of the nonlinear Kadomtsev-Petviashvili equation [53] (in the one-dimensional case, such an example is the Korteweg-de Vries equation). A discussion of general problems in the theory of solitons can be found, for example, in [54].

The source of standing excitations of the medium is given by the function $q(\boldsymbol{r}, t)$, which is the power density of the source of production of such excitation of the medium.

The spectral distribution of the probability of carrier's emission with frequency $\widetilde{\omega}$, normalized to the integral over frequency, is given by the function $P_{\widetilde{\omega}}(v)$. The frequency (energy) and velocity of carriers are determined by the law of motion of the carrier (for linear waves in a homogeneous stationary medium, this relation is the law of wave dispersion).

The reciprocal length of the free path of the carrier from the point of birth to the stop in the medium and transformation into a standing excitation of the medium is given by the "absorption coefficient" of the carriers with the frequency $\widetilde{\omega}$, $\kappa_{\widetilde{\omega}}(v)$.

The average lifetime of standing excitation is given by the parameter τ, and the average reciprocal time of the disappearance of standing fluctuations without the creation of a carrier of excitation (the so-called "quenching" of the excitation of the medium) is given by the parameter σ.

The processes of generation of standing and running excitations of the medium are elementary mechanical processes in the sense that they exist as mutually inverse mechanical (i.e. reversible in time) processes. The proposed model assumes that standing excitation forgets the history of its occurrence. Equations (10) and (11) are written for carriers with a certain velocity, and it is assumed that there is some velocity distribution, which, like all other parameters and functions introduced above, can be restored by comparing the predictions of this model with experimental data.

Note that in the case of an infinite velocity of carriers (i.e., in the case of transfer in the Lévy flights mode), the system of equations (10), (11) reduces to the Biberman-Holstein equation (2), and when this velocity is finite (i.e., in the case of transfer in the Lévy walk mode) instead of the function $f(\boldsymbol{r}_1,t)$ in (2) there is a function

$$\theta\left(t-\frac{|\boldsymbol{r}-\boldsymbol{r}_1|}{c}\right) f\left(\boldsymbol{r}_1, t-\frac{|\boldsymbol{r}-\boldsymbol{r}_1|}{c}\right)$$

(here $\theta(x)$ — the Heaviside step function), which corresponds to taking into account the retardation. The derivation of such a generalization of the Biberman-Holstein equation from the system of equations (10), (11) can be found, for example, in section 2 in [41].

An essential feature of the system of equations (10), (11) is that it is possible to reduce dependence of practically interesting integral characteristics only on the Holstein function. This means that such differential characteristics as the normalized spectrum of the source function, $P_{\widetilde{\omega}}(v)$, and the reciprocal free path, $\kappa_{\widetilde{\omega}}(v)$, enter the final result only as part of the Holstein function.

It concerns the law of motion of the excitation front from a point instantaneous source, i.e. effective front of the Green's function. In contrast to the case of Lévy flights, where one has to work with the definition of the front in the form (6), for standing (and similarly for running) excitations of the medium, this front can be specified by a simple standard relation

$$\left(r_{rest}(t)\right)^2 = \left(\int r^2 f_{rest}(\boldsymbol{r},t)d\boldsymbol{r}\right) / \left(\int f_{rest}(\boldsymbol{r},t)d\boldsymbol{r}\right), \quad (12)$$

since the numerator in (12) no longer diverges due to the limitation of distances by the ballistic motion front:

$$r_{ball} = vt. \quad (13)$$

For the front (12), as well as for the front of excitation carriers, in [43] for a wide class of distribution functions over the free path of carriers (3),

$$W_{step}(\rho) = \frac{\gamma\kappa_0}{(1+\kappa_0\rho)^{\gamma+1}}, \quad 0 < \gamma < 2, \quad (14)$$

where k_0 is an inverse characteristic length, which in the case of excitation transfer by resonant photons corresponds to the value of the absorption coefficient at the center of the spectral line, analytical expressions were obtained for practically interesting times and distances in transfer problems, namely for $\kappa_0\rho \gg 1$ and $t \gg \tau$.

It was shown in [12] that the proximity of the nonlocality parameter γ, obtained in [12], to the corresponding parameter for the Richardson t^3 law (1) make it possible to qualify the transport of density fluctuations in a tokamak plasma across a strong magnetic field as turbulence. Note that studies of the nonlocal properties of turbulence, including the deviation of statistics from the Gaussian one in various plasma turbulence phenomena, are reflected in the collective monograph [55].

The phenomenological model [12] of turbulence nonlocality, based on the system of kinetic equations (10), (11) and going beyond the diffusion Fokker-Planck models, can be considered, when applied to plasma, as a phenomenological generalization of the quasilinear theory of weak plasma turbulence, which goes back to [56], in the wave part of this kinetics (a review of the status of the quasilinear approach to plasma turbulence is presented in [57]). Going beyond the diffusion Fokker-Planck models has already been made, and for specific physical models in the case of a stationary flow in the space of wave numbers (not the thermodynamic limit), examples of the Kolmogorov spectrum in various problems of physics have been found [58]. In the problem considered in [12], it is important that it was possible to establish the closeness of the kinetics of plasma density fluctuations to what we call turbulence within the general framework of interpreting the results of experiments, without specifying the physical model of elementary excitations of the medium.

3. EXTENSION OF RICHARDSON'S T^3 LAW TO THE COMBINED LÉVY FLIGHT AND LÉVY WALK REGIME

The hypothesis of the locality of elementary processes in the existing quasi-linear theory of weak plasma turbulence seems to be quite justified, since for elementary processes it is possible to propose mechanical (i.e., reversible in time) models from first principles. For hydrodynamic turbulence, this aspect inevitably requires additional axiomatics, which obviously are the well-known hypotheses of Kolmogorov and Obukhov for homogeneous stationary turbulence. Lack of rigorous justification, i.e. the lack of a derivation from the Navier-Stokes equations sometimes allows researchers to qualify this approach as dimensional reasoning. Therefore, in the existing rather free field in the theory of hydrodynamic turbulence, it is quite legitimate to propose other models with their own axiomatics. Such an attempt is the generalization of Richardson's t^3 law (1) given below to the combined regime of Lévy flights and Lévy walks. Such a generalization is suggested, as noted above, by the idea of Schlesinger and colleagues [10] and the success of the model [12] in interpreting experiments on cross-correlation reflectometry of tokamak plasma.

Let us turn to model (10), (11) with the intention of establishing a connection between the phenomenological parameters introduced in it and the key parameters of the existing theory of hydrodynamic turbulence. It is important to note that although model (10), (11) did not discuss the possible physical mechanisms of elementary acts in the kinetic model, such possibilities in the theory of linear waves are well known and form the basis of the already mentioned quasilinear theory of weak plasma turbulence. The problem, however, lies in the fact that models of superdiffusion transfer of energy by linear plasma waves have not yet been created, which would provide explanations for the observed nonlocality phenomena, for example, in thermonuclear plasma (for example, we repeat the reference to [44, 45], but this list can be continued up to the present moment). Therefore, the problem of identifying adequate elementary acts responsible for the observed phenomena of superdiffusion transfer is to a comparable extent faced by both the theory of turbulent plasma and the theory of non-plasma hydrodynamic turbulence.

In this section, we will not solve the latter problem, but will only draw a bridge between the phenomenology of plasma and non-plasma hydrodynamic turbulence.

For hydrodynamic turbulence of fluids and gases, the key parameters are the Kolmogorov length η and velocity v_η, as well as the corresponding time t_η:

$$\eta = (\nu^3/\varepsilon_0)^{1/4},\ \ v_\eta = (\nu\varepsilon_0)^{1/4},\ \ t_\eta = (\nu/\varepsilon_0)^{1/2}, \tag{15}$$

where ν is the kinematic viscosity in units of m^2/s, ε_0 characterizes the specific energy dissipation rate in units of m^2/s^3.

Let us propose analogs of the Kolmogorov parameters in a kinetic transport model of the Biberman-Holstein type with allowance for retardation. An analogue of the Kolmogorov length is the path length at the center of the spectral line, k_0, since, as in the Kolmogorov model, is the minimum characteristic length:

$$\frac{1}{(\kappa_0)_{turb}} \sim \eta. \tag{16}$$

Of course, this and other parameter relationships make sense up to an order of magnitude.

An analogue of the Kolmogorov velocity is the characteristic velocity of running excitations of the medium (carriers). Batchelor's scaling for the initial stage of mutual separation of closely spaced test particles works in favor of the analogy for velocities:

$$\left(r_{pair}(t)\right)^2 \sim \left(v_\eta t\right)^2 \sim (vt)^2. \tag{17}$$

In the plasma turbulent motions along the minor radius of a toroidal plasma in a tokamak, this corresponds to a characteristic velocity restored [12] from the observed peaks in the spectrum of the quasi-coherent mode. Since t_η has a simple relationship with η and v_η, we can put

$$\tau \sim t_\eta. \tag{18}$$

To estimate the pair correlation function (turbulent relative dispersion) in order of magnitude, which is quite acceptable considering the very status of such an integral estimate of the distribution function in kinetic problems, we will use the analytical approximation results for the Green's function of the nonlocal transfer in the combined mode of Lévy flights and Lévy walks, according to (7.10) in [43]:

$$\left(\frac{r_{pair}(t, R_c, \gamma)}{\eta}\right)^2 \sim \frac{1}{\left(\frac{t_\eta}{t}\right)^{2/\gamma} + \left(\frac{t_\eta}{t}\right)^2 \frac{1}{{R_c}^2}\left(\frac{1+\gamma}{1-\gamma}\right)}, \tag{19}$$

where R_c is the retardation parameter,

$$R_c = c\tau\kappa_0, \tag{20}$$

which is the ratio of the lifetime of the excitation of the medium at rest and in motion (here c is the characteristic velocity of the carriers). With the specified correspondence to the Kolmogorov turbulence parameters (15), R_c turns out to be just a certain number in the range determined by the accuracy of estimates (16)-(18), i.e. actually turns out to be some free parameter. Preservation of the last factor in the second term in the denominator in (19) ensures that the front velocity (turbulent pair dispersion) is less than the ballistic limit (13).

Note that the proposed scaling (19) covers only the transition between transfers in the Lévy flight and Lévy walk modes. To construct a more general scaling, it is necessary to take into account the Batchelor ballistic regime at the initial stage and the diffusion regime at the final stage, see Figs. 5-7 in [59].

Calculation of (19) for γ=2/3 are shown in Fig. 1. If the retardation effect is neglected (i.e. $R_c \to \infty$), Richardson's t^3 law (1) is obtained from (19). Comparison of calculations for various values of the retardation parameter with the results of numerical simulations in [59], where Richardson scaling (1) works up to time $t{\sim}10^2 t_\eta$, shows that a possible niche exists for taking into account the retardation and the respective appearance of ballistic scaling of Lévy walks with a value of the retardation parameter R_c equal to few-several tens.

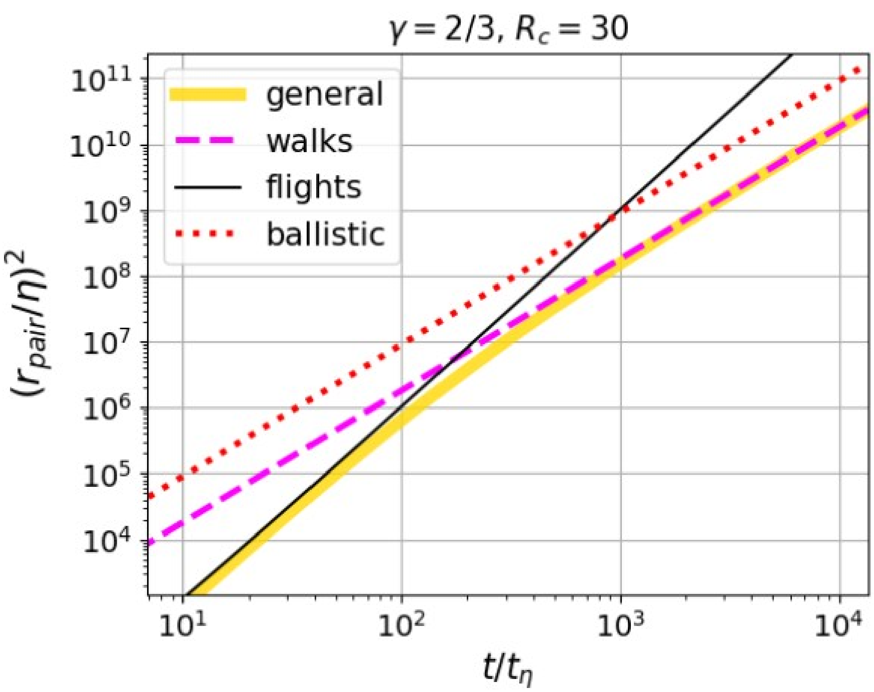

Figure 1. Dependence of pair dispersion on time for the nonlocality parameter γ=2/3 and retardation parameter R_c = 30. The results are shown for Lévy flights (black line), Lévy walks (pink dashed line), combined Lévy flights and walks (19) (yellow curve); ballistic front (13) (red dotted line).

The faster growth of the pair correlation with time, obtained in simulations [59] in the regions where the Richardson law (1) was expected (i.e., between the Batchelor ballistic regime and the diffusion regime, see Fig. 7 in [59]), suggests that the case of smaller values of the nonlocality parameter γ may also be considered, for example, $\gamma = 1/2$. This choice is also interesting in that the spectral probability density shape $P_{\tilde{\omega}}$ corresponding to this case in model (10),(11) has a Lorentzian form, which often occurs in various physical models, where the broadening of the spectral distribution compared to the monochromatic one is due to the finiteness of the lifetime of the excited state, the relaxation of which leads to the birth of a carrier (running excitation of the medium).

Calculations by formula (19) for $\gamma = 1/2$. are shown in fig. 2. Comparison for various values of the retardation parameter with the results of numerical simulations in [59] shows that a possible niche for taking into account the retardation and the respective appearance of the ballistic scaling of Lévy walks exists for the retardation parameter R_c of the order of 10^2.

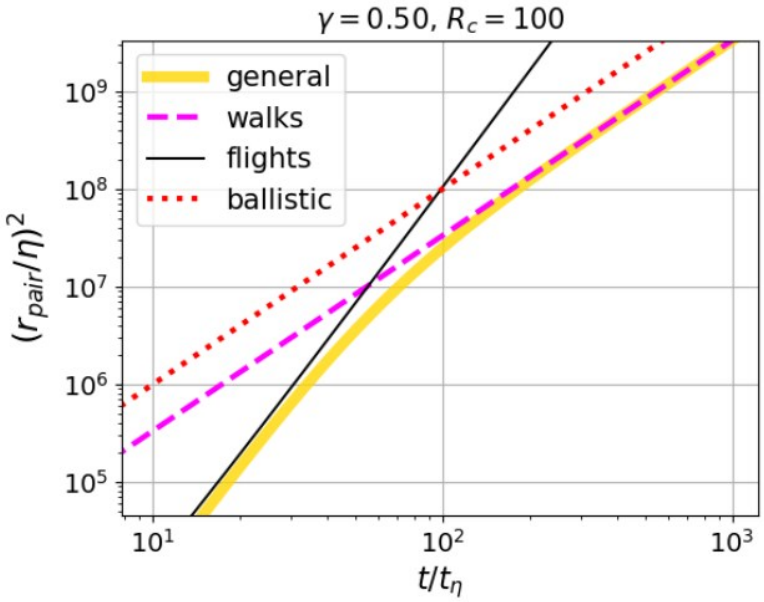

Figure 2. Dependence of pair dispersion on time for the nonlocality parameter γ=1/2 and retardation parameter R_c = 100. The results are shown for Lévy flights (black line), Lévy walks (pink dashed line), combined Lévy flights and walks (19) (yellow curve); ballistic front (13) (red dotted line).

Thus, the kinetic model (10), (11) allows us to qualitatively consider the problem of expanding the range of applicability of the Richardson law (1) and the problem of its possible generalization and reassessment, which, in particular, was actively discussed in [3]. Although Richardson's law (1) is supported by an extensive database, including, for example, recent experimental and theoretical studies in [60], the behavior of turbulent pair correlation (turbulent pair dispersion) at large times and the change of regimes (scalings) with time is of undoubted interest.

4. CONCLUSIONS

In this paper, we show the possibility of a universal description of the characteristics of nonlocality of transfer in a stochastic medium (including turbulence of gases and fluids) using a formalism like the Biberman-Holstein model for the transfer of excitation of a medium by photons, generalized to take into account the finiteness of the velocity of excitation carriers. Universality lies primarily in the fact that the main results are explicitly expressed in terms of the distribution function of medium excitation carriers in their free path, which has the form of a Lévy distribution, and such a function is directly related to the Holstein function in the Biberman-Holstein model. The approach we developed earlier [12] made it possible to establish the closeness of the nonlocality (superdiffusion) parameter of plasma density fluctuations moving across a strong magnetic field in a tokamak to the Richardson t^3 law for the mean square separation of a pair of particles in a fluid or gaseous medium. The key feature of the developed approach is that it was possible to establish the proximity of the kinetics of plasma density fluctuations to hydrodynamic turbulence within the general framework of interpreting the results of experiments using the phenomenological model of transfer in the Lévy walk mode, which does not require specifying the physical model of elementary excitations of the medium.

The developed kinetic model made it possible to suggest at a qualitative level a generalization of Richardson's t^3 law for the combined regime of Lévy flights and Lévy walks for the turbulence in fluids and gases. Although Richardson's t^3 law is supported by an extensive database, the behavior of turbulent pair correlation (turbulent pair dispersion) at large times and the change in regimes (scalings) with time is of undoubted interest.

Acknowledgements

This work has been carried out using computing resources of the federal collective usage center Complex for Simulation and Data Processing for Mega-science Facilities at NRC "Kurchatov Institute"; http://ckp.nrcki.ru/.

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