IRAMP
15(1), Jan 2024, pp. 39-43

ISSN: 2229-39-44

Singular Attractive Potentials can have Zero Energy Bound States

EUGENE OKS*

Physics Department, 380 Duncan Drive, Auburn University, Auburn, AL 36849, USA
**Corresponding author email: oksevgu@auburn.edu*

ABSTRACT: Potentials of the form $V(r) = -a_n/r^n$, a_n being a positive or negative constant and $n > 2$ being the integer, have been the subject of various studies for many years. Studying these singular potentials is important because they correspond to a large number of physical systems. The focus of the present paper is on the zero energy states of attractive singular potentials. The existing paradigm is that for negative interaction potentials that vanish as r increases toward infinity, the existence of bound states is possible only for the negative total energy: both classically and quantally, bound states of the zero energy deemed impossible. The first attempt to break this paradigm was made in one of our previous papers for the interaction potential $V(r) = -a_3/r^3$, where $a_3 > 0$. Two specific examples were neutron-neutron systems and neutron-muon systems for the configuration of the parallel magnetic dipole moments of the two particles in the pair. The existence of zero energy bound states was shown in that paper both for the neutron-neutron system ("neutronium") and for the neutron-muon system ("neutron-muonic atom"). In the present paper we demonstrate the existence of the bound states of zero energy for *all* attractive singular potentials $V(r) = -a_n/r^n$, where $a_n > 0$, for $n > 2$. We prove that the corresponding normalization integrals converge and then we actually calculate them explicitly. The final result is the explicit form of the corresponding normalized wave functions of the bound states of zero energy.

Keywords: singular attractive potentials; zero energy bound states; van der Waals interaction; Casimir-Polder retarded regime

1. INTRODUCTION

Potentials having the form

$$V(r) = -a_n/r^n, \tag{1}$$

a_n being a positive or negative constant and $n > 2$ being the integer, have been the subject of various studies for many years – see, for example, papers [1-12] and references therein. Studying these singular potentials is important because they correspond to a large number of physical systems, as specified below.

The case of $n = 3$. This case corresponds to two identical atoms in two different states – see, for instance, textbook [13], Eq. (86.3). The unperturbed system (that is, two isolated atoms) is invariant with respect to the interchange, so that there is an additional degeneracy. In the situation where the two atoms have different parity, and their angular momenta either differ by ±1 or are equal to each other (but are not zeros), there are nonzero nondiagonal matrix elements of the dipole moment operator. Therefore, the processing of the dipole term in the perturbation operator through the secular equation yields the interaction potential

$$V(r) = \text{const}/r^3. \tag{2}$$

The same interaction potential results also from the tensor force between nucleons – see, for instance, work [10].

The case of $n = 4$. This case corresponds to the interaction between a charge and an induced dipole, such as when electrons or ions move through a gas of molecules of a relatively small size. – see, for example, paper [4]. It also corresponds to the interaction of a neutral atom with perfectly conducting wall, as presented in paper [2]. In both subcases, the result is the attractive interaction potential of the form

$$V(r) = -a_4/r^4, \quad a > 0. \tag{3}$$

The case of n = 5. This case corresponds to two atoms in the states of nonzero orbital and total angular momenta – see, for instance, textbook [13], Eq. (86.2). While the average dipole moment in these states is zero, the average quadrupole moment is not zero. Therefore, the quadrupole-quadrupole interaction calculated in the first order of the perturbation theory yields the interaction potential

$$V(r) = \text{const}/r^5. \tag{4}$$

The same interaction potential results also from a perturbative correction to the tensor force in the nuclear potential – see, for example, work [10].

The case of n = 6. This case corresponds to the "standard" van der Waals interaction between two atoms in the ground states resulting in the attractive potential (see, for instance, textbook [13], Eq. (86.1))

$$V(r) = -a_6/r^6, \qquad a_6 > 0. \tag{5}$$

The case of n = 7. This case corresponds to the van der Waals forces in the retarded regime (Casimir-Polder regime) yielding the attractive potential [2]:

$$V(r) = -a_7/r^7, \qquad a_7 > 0. \tag{6}$$

The focus of the present paper is on the zero energy states of attractive singular potentials. The existing paradigm is that for negative interaction potentials that vanish as r increases toward infinity, the existence of bound states is possible only for the negative total energy E: both classically and quantally, bound states of E = 0 deemed impossible. For instance, textbook [13] in its Sect. 18 said that for potentials behaving at large r as $-1/r^s$ where s > 2 (that is, for the attractive singular potentials that are the focus of the present paper), the highest discrete energy level is characterized by a *nonzero negative* value E_{max}, so that states of larger energies, including E = 0, cannot be the bound states.

The first attempt to break this paradigm was made in paper [14] for the interaction potential given by Eq. (2). Two specific examples in paper [14] were neutron-neutron systems and neutron-muon systems for the configuration of the parallel magnetic dipole moments of the two particles in the pair. The existence of zero energy bound states was shown in paper [14] both for the neutron-neutron system ("neutronium") and for the neutron-muon system ("neutron-muonic atom").

In the present paper we demonstrate the existence of the bound states of zero energy for *all* attractive singular potentials $V(r) = -a_n/r^n$, where $a_n > 0$, for n > 2. We prove that the corresponding normalization integrals converge and then we actually calculate them explicitly. The final result is the explicit form of the corresponding normalized wave functions of the bound states of zero energy.

2. THE PROOF OF THE EXISTENCE OF THE ZERO ORDER BOUND STATES FOR ALL ATTRACTIVE SINGULAR POTENTIALS

We consider two particles, whose reduced mass is m, described by the attractive singular interaction potential from Eq. (1) with $a_n > 0$. For zero total energy, the Schrödinger equation for the radial wave function R(r), after the substitution

$$R(r) = u(r)/r, \tag{7}$$

can be represented in the form

$$d^2u/dr^2 - [L(L + 1)/r^2]u + (b/r^n)u = 0, \tag{8}$$

where

$$b = 2a_n m/(\hbar^2). \tag{9}$$

At relatively small r, Eq. (8) can be approximated as

$$d^2u/dr^2 \approx -bu/r^n. \tag{10}$$

We seek the solution of Eq. (10) as follows:

$$u(r) = \text{const}\cos(a/r^b). \tag{11}$$

On substituting Eq. (11) in Eq. (10), we get

$$-\{[cd(d+1)/r^{d+2}]\sin(c/r^d) + [c^2d^2/r^{2d+2}]\}u \approx -(b/r^n)u. \tag{12}$$

At small r, the second term in braces dominates, so that Eq. (12) simplifies to

$$-(c^2d^2/r^{2d+2})u \approx -(b/r^n)u. \tag{13}$$

From Eq. (13) we find:

$$d = n/2 - 1, \qquad c = b^{1/2}/(n/2 - 1). \tag{14}$$

Thus, for relatively small r, we obtained

$$u(r) \approx \text{const}\cos\{b^{1/2}/[(n/2-1)r^{n/2-1}]\}. \tag{15}$$

At relatively large r, Eq. (8) can be reduced to:

$$d^2u/dr^2 \approx [L(L+1)/r^2]u. \tag{16}$$

We seek the solution of Eq. (16) as follows:

$$\chi(r) = \text{const}/r^q. \tag{17}$$

After substituting Eq. (17) in Eq. (16), we obtain

$$q(q+1)/r^{q+2} \approx [L(L+1)/r^{q+2}. \tag{18}$$

From Eq. (18) it is see that

$$q = L \tag{19}$$

(the second solution q = –L – 1 does not have a physical meaning). Thus, at relatively large r, the solution of Eq. (8) has the form:

$$u(r) \approx \text{const}/r^L. \tag{20}$$

From the asymptotics, expressed by Eqs. (15) and (20), one can see that the normalization integral

$$\int_0^\infty dr\,[u(r)]^2 \tag{21}$$

converges for any $L \geq 1$. Two additional comments are in order.

First, the root mean square value of r and the average value of r exist only for $L \geq 2$. Second, for a pair of identical particles, the states of the system are characterized by a definite parity $P = (-1)^L$ (due to the invariance with respect to the inversion of the reference frame – see, e.g., the textbook [Landau]). For example, for two neutrons in the state of the parallel spins and thus parallel magnetic moments, considered in paper [14]), the coordinate wave function should be antisymmetric: consequently, the parity P = –1. In this satiation, only the states characterized by odd values of L are physically admissible. In view of the above restriction $L \geq 2$, this leads to the conclusion that the lowest physically admissible value of the angular momentum for such system is L = 3.

The intermediate summary: systems interacting by the attractive singular potential $V(r) = -a_n/r^n$, where $a_n > 0$, have zero energy bound states for any $n > 2$. This fundamental result breaks the paradigm that such bound states cannot exist.

3. NORMALIZED WAVE FUNCTIONS OF THE ZERO ENERGY BOUND STATES

For $L = 0$, the following unnormalized wave function u(r) for the zero energy state in the *attractive* singular potentials $V(r) = -a_n/r^n$, where $a_n > 0$, was presented in paper [9]:

$$u(r) = r^{1/2} J_{1/(n-2)}[a_n^{1/2} r^{1-n/2} /(1-n/2)], \tag{22}$$

where $J_g[f(r)]$ is the Bessel function. For arbitrary L, the following unnormalized wave function u(r) for the zero energy state in the *repulsive* singular potentials $V(r) = -a_n/r^n$, where $a_n < 0$, was presented in paper [5]:

$$u_{repuls}(r) = r^{1/2} K_{1/(n-2)}[a_n^{1/2} r^{1-n/2} /(n/2-1)], \tag{23}$$

where $K_g[f(r)]$ is the modified Bessel function of the second kind. Based on these results, we represent the unnormalized wave function u(r) for the zero energy state in the *attractive* singular potentials *for arbitrary angular momentum L* as follows:

$$u(r) = r^{1/2} J_{(2L+1)/(n-2)}[a_n^{1/2} r^{1-n/2}/(1-n/2)]. \tag{24}$$

Now we calculate the Normalization Integral (NI)

$$NI = \int_0^\infty dr\, r\{J_{(2L+1)/(n-2)}[a_n^{1/2} r^{1-n/2}/(1-n/2)]\}^2. \tag{25}$$

The results of the calculations are presented in Table 1 for $n = 3 - 7$ and $L = 1 - 3$.

Table 1. Normalizing Integrals (NI) for n = 3 – 7 and L = 1 – 3.

	n=3	n=4	n=5	n=6	n=7
L=1	0.050	0.133	0.200	0.255	0.300
L=2	0.00238	0.0190	0.0405	0.0606	0.0783
L=3	0.000397	0.00635	0.0169	0.0283	0.0389

From Table 1 it is seen that the NI increases with the growth of n, but decreases with growth of L. The normalized radial wave function of the zero energy state corresponding to attractive singular potentials is

$$R_{norm}(r) = r^{-1/2} J_{(2L+1)/(n-2)}[a_n^{1/2} r^{1-n/2}/(1-n/2)]/[NI(n, L)]^{1/2}. \tag{26}$$

It corresponds to the zero energy *bound state*.

4. CONCLUSIONS

We considered the zero energy states of attractive singular potentials $V(r) = -a_n/r^n$ ($a_n > 0$), for $n > 2$. We proved that the normalization integrals for the corresponding unnormalized wave functions converge for any n if the angular momentum satisfies the condition $L \geq 1$. So, lots of physical systems, characterized by singular attractive potentials with various values of n, have *bound states of zero energy*. Thus, we broke the paradigm that bound states of zero energy are impossible. This is the *fundamental result* in its own right.

We explicitly calculated the normalization integrals and illustrated the results both as the table and pictorially. We presented the explicit form of the corresponding normalized wave functions of the bound states of zero energy.

References

1. Margenau, H.: Van der Waals forces. Rev. Mod. Phys. 11, 1-35 (1939)
2. Casimir, H.B.G., Polder, D.: The influence of retardation on the London-van der Waals forces. Phys. Rev. 73, 360-372 (1948)

3. Case, K.M.: Singular potentials. Phys. Rev. 80, 797-806 (1950)
4. Vogt, E., Wannier, G.H.: Scattering of ions by polarization forces. Phys. Rev. 95, 1190-1198 (1954)
5. Khuri, N.N., Pais, A.: Singular potentials and peratization. I. Rev. Mod. Phys. 36, 590-595 (1964)
6. Frank, W.M., Land, D.J., Spector, R.M.: Singular potentials. Rev. Mod. Phys. 43, 36-98 (1971)
7. Gao, B.: Solutions of the Schrödinger equation for an attractive $1/r^6$ potential. Phys. Rev. A 58, 1728-1734 (1998)
8. Marinescu, M., Sadeghpour, H.R.: Long range potentials for two-species alkali-metal atoms. Phys. Rev. A 59, 390-404 (1999)
9. Beane, S.R., Bedaque, P.F., Childress, L., Kryjevski, A., McGuire, J., van Kolck, U.: Singular potentials and limit cycles. Phys. Rev. A. 64, 042103 (2001)
10. Beane, S. R., Bedaque, P. F., Haxton, W. C., Phillips, D. R., Savage, M. J.: From hadrons to nuclei: Crossing the border. In: At The Frontier of Particle Physics: Handbook of QCD, pp. 133-269. World Scientific, Singapore (2001)
11. Arnecke, F., Madroñero, J., Friedrich, H.: Jost functions and singular attractive potentials. Phys. Rev. A 77, 022711 (2008)
12. Milton, K.A. (2011). Resource letter VWCPF-1: van der Waals and Casimir–Polder forces. Amer. J. Phys. 79, 697-711 (2011)
13. Landau, L.D., Lifshitz, E.M.: Quantum Mechanics. Pergamon, Oxford (1965)
14. Oks, E.: Zero-energy bound states of neutron-neutron or neutron-muon systems. Foundations 3, 65-71 (2023)