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RESEARCH ARTICLE

APPLICATION OF THE GENERALIZED HAMILTONIAN DYNAMICS TO A MODIFIED COULOMB POTENTIAL

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Abstract: Dirac's Generalized Hamiltonian Dynamics (GHD), a purely classical formalism, is applied to spinless particles under the influence of a binomial potential. The integrals of the motion for this potential are chosen as the constraints of GHD and Fradkin's unit Runge vector is used in place of the Laplace-Runge-Lenz vector. A functional form of the unit Runge vector is derived for the binomial potential. It is shown — in accordance with Oks and Uzer (2002) — that there occurs a new kind of time dilation leading to classical stable, nonradiating states. The energy of these classical stable states agrees exactly with the corresponding quantal results for the ground state and for all states of odd values of the radial and angular harmonic numbers. The primary application of the obtained results is to pionic (and kaonic) atoms. Other applications include nanoplasmas and the precession of planetary orbits.

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1. INTRODUCTION

In 1950, Dirac developed a generalized Hamiltonian dynamics (hereafter GHD) [1–3]. The conventional Hamiltonian dynamics is based on the assumption that the momenta are independent functions of velocities. Dirac analyzed a more general situation where momenta are not independent functions of velocities [1–3]. Physically, the GHD is a purely classical formalism for constrained systems; it incorporates the constraints into the Hamiltonian. Dirac designed the GHD with applications to quantum field theory in mind [3].

The present work, where GHD is applied to atomic and molecular systems by choosing integrals of the motion as the constraints of the system, stems from a paper in which this idea was applied to hydrogenic atoms treated non-relativistically on the basis of the Coulomb potential [4]. Using this purely classical formalism, Oks and Uzer demonstrated the existence of non-radiating states and found their energy to be in exact agreement with the corresponding results of quantum mechanics. They employed two fundamental experimental facts, but did not "forcefully" quantize any physical quantity describing the atom. In particular, this amounted to classically deriving Bohr's postulate on the quantization of the angular momentum rather than accepting it on an axiomatic basis.

It important to point out that the physics behind classical non-radiating states is a new kind of time-dilation found by Oks and Uzer [4]. This is a non-Einsteinian time-dilation.

The subject of the present paper differs from the above mentioned paper by Oks and Uzer in that the dynamics analyzed are of a more general nature: a term proportional to $1/r^2$ is added to the Coulomb potential. This more complicated potential we call here the *binomial potential*. Then the generalized unit Laplace-Runge-Lenz vector [5, 6], or as named by Fradkin, the unit Runge vector [5], is utilized instead of the classical Laplace-Runge-Lenz vector.

This binomial potential has interesting applications. The primary application considered here is to pionic (and kaonic) atoms. We will classically obtain results corresponding to the solution of the quantal (relativistic) Klein-Gordon equation, the latter being appropriate because pions are spinless particles. Another application concerns the precession of planetary orbits: for this phenomenon Einstein's equations of general relativity are

equivalent to non-relativistic equations for the motion in the binomial potential [7]. We shall also briefly mention an application furnished by the description of the energy of nonradiating states of the so-called nanoplasmas [14].

The present paper has the following structure. In Section 2, we briefly outline Dirac's GHD. Section 3 serves to describe with more detail the applications of the binomial potential mentioned in the above paragraph. In Sections 4 and 5 we discuss the dynamical symmetries or Fradkin and the generalization of the Laplace-Runge-Lenz vector.

We present our new results in Section 6 and appendices *A*, *B*, and *C*. Section 7 contains the appendices.

2. DIRAC'S GENERALIZED HAMILTONIAN DYNAMICS (GHD)

Dirac [1–3] considered a dynamical system of N degrees of freedom characterized by generalized coordinates q_n and velocities $v_n = \frac{dq_n}{dt}$, where n = 1, 2, ..., N. If the Lagrangian of the system is

$$L = L(q, v), \qquad \ldots (2.1)$$

then momenta are defined as

$$p_n = \frac{\partial L}{\partial v_n} \qquad \ldots (2.2)$$

Each of the quantities q_n, v_n, p_n can be varied by δq_n, δv_n, δp_n, respectively. The latter small quantities are of the order of ε, the variation being worked to the accuracy of ε. As a result of the variation, Eq. (2.2) would not be satisfied any more, since their right-hand side would differ from the corresponding left side by a quantity of the order of ε. Indeed, since the Lagrange and Hamilton functions are related as $L = p_n v_n - H$, then for an arbitrary variation in the momenta one has (here and below the summation over a twice repeated suffix is understood):

$$\delta L = \left(v_n - \frac{\partial H}{\partial p_n} \right) \delta p_n = (v_n - v_n)\, \delta p_n = 0$$

In the above, we also used Hamilton's equation $v_n = \frac{\partial H}{\partial p_n}$.

Further, Dirac distinguished between two types of equations. To one type belong equations such as Eq. (2.2), which does not hold after the variation (he called them "weak" equations). In what follows, for weak equations, adopting Dirac's nomenclature, we use a different equality sign ≈ from the usual. Another type constitute equations such as Eq. (2.1), which holds exactly even after the variation (he called them "strong" equations).

If quantities $\partial L/\partial v_n$ are *not* independent functions of velocities, one can exclude velocities v_n from Eqs (2.2) and obtain one or several weak equations

$$\phi(q, p) \approx 0, \qquad \ldots (2.3)$$

containing only q and p. In his formalism, Dirac [1–3] used the following complete system of independent equations of the type (3):

$$\phi_m(q, p) \approx 0, \quad (m = 1, 2, \ldots M). \ldots (2.4)$$

Here the word "independent" means that neither of the ϕ's can be expressed as a linear combination of the other ϕ's with coefficient depending on q and p. The word "complete" means that any function of q and p, which would become zero allowing for Eq. (2.2) and which would change by ε under the variation, should be a linear combination of the functions $\phi_m(q, p)$ from (4) with coefficients depending on q and p.

Finally, proceeding from the Lagrangian to a Hamiltonian, Dirac [1–3] obtained the following central result:

$$H_g = H(q, p) + u_m \phi_m(q, p) \qquad \ldots (2.5)$$

Equation (2.5) is a strong equation expressing a relation between the generalized Hamil-tonian H_g and the conventional Hamiltonian $H(q, p)$. Quantities u_m are coefficients to be determined. Generally, they are functions of q, v, and p; by using Eq. (2.2), they could be made functions of q and p. It should be emphasized that $H_g \approx H(q, p)$ would be only a weak equation—in distinction to Eq. (2.5).

Equation (2.5) shows that the Hamiltonian is not uniquely determined, because a linear combination of ϕ's may be added to it. Equation (2.4) are called constraints. The above distinction between constraints (i.e., weak equations) and strong equations can be reformulated as follows.

Constraints must be employed in accordance to certain rules. Constraints can be added. Constraints can be multiplied by factors (depending on q and p), but only on the left side, so that these factors must not be used inside Poisson brackets.

If f is some function of q and p, then $\frac{df}{dt}$ (i.e., a general equation of motion) in the Dirac's GHD is

$$\frac{df}{dt} = [f, H] + u_m [f, \phi_m], \qquad \text{... (2.6)}$$

where $[f, g]$ is the Poisson bracket defined for two functions f and g of the canonical variables p and q as:

$$[f, g] = \frac{\partial f}{\partial q_r}\frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial g}{\partial q_r}. \qquad \text{... (2.7)}$$

where r is an index put to stress the fact that in general there will be several generalized coordinates and momenta. Substituting ϕ'_m in (2.6) instead of f and taking into account Eq. (2.4), one obtains:

$$[\phi_{m'}, H] + u_m [\phi_{m'}, \phi_m] \approx 0. (m' = 1, 2, \ldots, M). \qquad \text{... (2.8)}$$

These consistency conditions allow determining the coefficients u_m.

3. APPLICATIONS OF THE BINOMIAL POTENTIAL

3.1 Pionic Atoms Described by the Klein-Gordon Equation of Relativistic Quantum Mechanics

Relativistic treatments of the hydrogenic atoms are typically presented working with the Dirac equation, which is a relativistic wave equation for spin-1/2 particles. However, in the literature one can also find a treatment of *relativistic* hydrogenic atoms ignoring spin; that is, working with the Klein-Gordon equation (hereafter, the KG equation) [8, 10–13].

The radial KG equation for the problem of the hydrogenic atom is given by:

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[\frac{\lambda}{\rho} - \frac{1}{4} - \frac{l(l+1) - (Z\alpha)^2}{\rho^2}\right] R = 0 \cdot \qquad \text{... (3.1)}$$

where Z is the nuclear charge and $\alpha = \frac{e^2}{c\hbar} \cong \frac{1}{137}$ is the fine structure constant. Other notations in Eq. (3.1) are as follows:

$$\lambda = Z\alpha E/(M^2c^4 - E^2)^{1/2}, \quad \rho = \beta r,$$
$$\beta = 2\ (M^2c^4 - E^2)1/2/(\hbar c).$$

The radial KG Eq. (3.1) for the Coulomb potential is equivalent to the radial Schrödinger equation with a potential U and an energy W, such that

$$U/(4W) = \lambda/\rho + (Z\alpha)^2/\rho^2, \qquad (W < 0),$$

which is a binomial potential.

For usual hydrogenic atoms, the fine structure splitting predicted by the KG equation is greater than what is observed experimentally [8]. However, for *pionic* (and *kaonic*) atoms, the KG equation becomes *exact* assuming the nucleus to be point-like. Indeed, the pionic atom is an exotic hydrogenic atom, where the atomic electron is substituted by a negative pion, which is *spinless*. Negative pions are *spinless* particles of the same charge as electrons, but 273 times heavier than electrons.

3.2 Precession of Planetary Orbits

In his seminal paper, *Die Grundlange der allgemeinen Relativitästhoerie* [7], Einstein showed that general relativistic effects perturb the Kepler potential by an additive term proportional to $1/r^2$ and used it to calculate the precession of Mercury's orbit around the sun. His calculations for the precession yielded 43"/century, which was later confirmed by observations. There is a number of textbooks on general relativity presenting this result [15–17].

3.3 Radiation of Nonrelativistic Particles in a Central Field

Karnakov et al. [14] derive the spectrum and expressions for the intensity of dipole radiation for a classical nonrelativistic particle executing nonperiodic motion. The potential in which the particles under consideration move is of the form $U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}$. The authors of this paper apply their results to the description of the radiation and the absorption of a classical collisionless electron plasma in nanoparticles irradiated by an intense laser field. Also, they find the rate of collisionless absorption of electromagnetic wave energy in equilibrium isotropic nanoplasma.

4. DYNAMICAL SYMMETRIES BY FRADKIN

Fradkin [5] has shown that all classical dynamical problems of both the relativistic and non-relativistic type,

dealing with a central potential, necessarily possess O (4) and SU (3) symmetries. This led him to a generalization of the Runge-Lenz vector in the Kepler problem. Here we will briefly present his results relating to the generalization of the Runge-Lenz vector and the construction of the elements of the Lie algebra of O (4) and SU (3) in terms of canonical variables.

In the non-relativistic Kepler problem the force on the affected particle is an inverse square force given by:

$$\dot{\mathbf{p}} = -\frac{\lambda}{r^2}\hat{r};\ \mathbf{p} = m\dot{\mathbf{r}},\ \hat{r} = \frac{\mathbf{r}}{r} \qquad \ldots (4.1)$$

and the overdot denotes total differentiation with respect to time. In the Kepler problem, the Hamiltonian and the angular momentum vector **L** are the conserved quantities. There also exists another conserved vector quantity, namely the Laplace-Runge-Lenz vector, or simply the Runge-Lenz vector defined as

$$\mathbf{A} = \left(-2mE\right)^{-\frac{1}{2}}\left(\mathbf{p}\times\mathbf{L} - \lambda m\hat{r}\right) \qquad \ldots (4.2)$$

This vector lies in the plane of the orbit and points from the nucleus to the perihelion of the orbit; some authors refer to it as the eccentricity vector [10], as shown in Fig. 1.

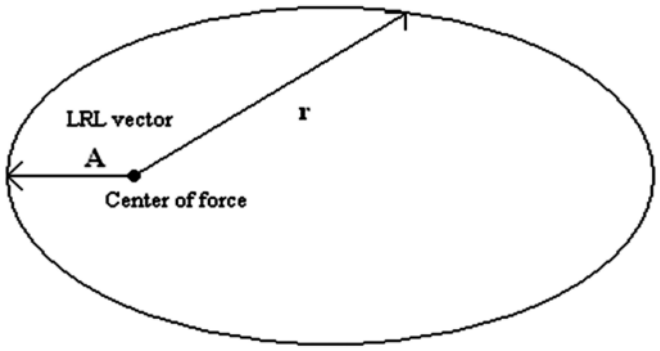

Fig. 1: Direction of the Laplace-Runge-Lenz (LRL) vector **A** within the elliptic trajectory, corresponding to the motion in a Coulomb field or a Kepler-Newton field

Fradkin found, by differentiation via the standard Poisson bracket formalism, that for the Kepler problem, and indeed for all central potential problems, that **A**, **L**, and H satisfy the following closed Lie algebra:

$$\left[A_i, H\right] = \left[L_i, H\right] = 0$$

$$\left[L_i, L_j\right] = \varepsilon_{ijk}L_k$$

$$\left[L_i, A_j\right] = \varepsilon_{ijk}A_k$$

$$\left[A_i, A_j\right] = \varepsilon_{ijk}L_k \qquad \ldots (4.3)$$

It is seen that the Lie algebra given above is isomorphic to that of the generator of the O (4) symmetry group, which is the group of orthogonal transformations representing rotations in four dimensions. Fradkin also concluded that if the existence of the Runge-Lenz vector is simply to ensure that the plane of the motion is conserved, then it should always be possible to find a vector analogous to the Runge-Lenz vector for all central potentials.

Fradkin proposed a generalization for the Runge-Lenz vector by choosing $\hat{r}$, $\hat{L}$, and $\hat{r}\times\hat{L}$ as a mutually orthogonal triad of unit vectors. This unit Runge vector is

$$\hat{k} = (\hat{k}\cdot\hat{r})\,\hat{r} + (\hat{k}\cdot\hat{L})\,\hat{L} + (\hat{k}\cdot\hat{r}\times\hat{L})\,\hat{r}\times\hat{L}, \qquad \ldots (4.4)$$

but since the unit Runge vector is in the plane of the orbit and the angular momentum vector is perpendicular to the plane of motion, then the second term is identically zero, and $\left(\hat{k}\perp\mathbf{L}\right)\cdot\hat{k}$ may be chosen to be the direction from which the azimuthal angle θ is measured (with the positive sign given by a right-handed rotation about $\hat{L}$), then we have:

$$\hat{r}\cdot\hat{k} = \cos\theta \quad \text{and} \quad \hat{k}\cdot\hat{r}\times\hat{L} = \sin\theta \qquad \ldots (4.5)$$

thus

$$\hat{k} = (\cos\theta)\,\hat{r} + (\sin\theta)\,\hat{r}\times\hat{L} \qquad \ldots (4.6)$$

Defining $u = 1/r$, we may write the following differential equation for u and the azimuthal angle θ in terms of the energy E, potential V and angular momentum L:

$$\left(\frac{du}{d\theta}\right)^2 = \left(\frac{2m}{L^2}\right)(E - V) - u^2 \qquad \ldots (4.7)$$

At this point we note the following relations and definition:

$$\cos\theta = f\,(u, L^2, E)$$

$$\sin\theta = \left(\frac{\partial f}{\partial u}\right)\frac{(\hat{r}\cdot\mathbf{p})}{L} \qquad \ldots (4.8)$$

Further, after putting $V = -\lambda u$ for the potential of the Kepler problem, the orbit equation becomes:

$$f = \cos\theta = \left[2mEL^2 + (\lambda m)^2\right]^{-1/2}\left(L^2 u - \lambda m\right). \quad ...(4.9)$$

The unit Runge vector may be expressed as:

$$\hat{k} = \left[f - u\frac{\partial f}{\partial u}\right]\hat{r} + L^{-2}\frac{\partial f}{\partial u}\mathbf{p}\times\mathbf{L} \quad ... (4.10)$$

Its Poisson bracket with the Hamiltonian function vanishes. Poisson brackets containing between the components of the unit Runge vector also vanish. A complete set of Poisson brackets involving the unit Runge vector is the following:

$$\left[\hat{k}_i, H\right] = 0; \quad \left[\hat{k}_i, \hat{k}_j\right] = 0;$$
$$\left[L_i, \hat{k}_j\right] = \varepsilon_{ijk}\hat{k}_k; \text{ for } i, j, k = 1, 2, 3. \quad ... (4.11)$$

5. FURTHER RESULTS ON THE GENERALIZATION OF THE LAPLACE-RUNGE-LENZ VECTOR

Holas and March [6] provided a further development of the unit Runge vector. They focused on the construction and time dependence of the vector itself rather than on the dynamical symmetries of central potentials or the algebras satisfied by the unit Runge vector.

Holas and March used the relation

$$\mathbf{p}\times\mathbf{L} = \frac{\mathbf{r}L^2}{r^2} - \frac{(\mathbf{p}\cdot\mathbf{r})}{r^2 L}\mathbf{L}\times\mathbf{r} \quad ... (5.1)$$

to rewrite the unit Runge vector, Eq. (4.10), as:

$$\hat{k} = f\,\hat{r} - \frac{(\mathbf{p}\cdot\mathbf{r})}{Lr}\frac{\partial f}{\partial u}\hat{L}\times\hat{r} \quad ... (5.2)$$

where the function f is specified in the next section. This is the form of the unit Runge vector with which we shall work.

6. APPLICATION OF THE GENERALIZED HAMILTONIAN DYNAMICS TO THE BINOMIAL POTENTIAL

In our case, the angular momentum vector and the unit Runge vector are constants of the motion for a centrally symmetric potential and consequently have vanishing Poisson brackets with the Hamiltonian for the system and are thus suitable constraints for the application of GHD. Following Oks and Uzer [4], the Hamiltonian for this system is:

$$H_g = \frac{p^2}{2\mu} - \frac{Ze^2}{r} + \frac{\Lambda}{2\mu r^2} + \mathbf{u}\cdot(\mathbf{L} - \mathbf{L}_0) + \mathbf{w}\cdot\left(\hat{k} - \hat{k}_0\right), \quad ... (6.1)$$

where Λ is the strength of the binomial potential, Ze is the nuclear charge, $-e$ is the electron charge, μ is the reduced mass, $\mathbf{u}$ and $\mathbf{w}$ are yet unknown constant vectors (to be determined later) of the GHD formalism, $\mathbf{L}_0$ and $\hat{k}_0$ are the values of the angular momentum and unit Runge vector in a particular state of the motion so that in those states

$$\mathbf{L} \approx \mathbf{L}_0 \quad ... (6.2)$$

and

$$\hat{k} \approx \hat{k}_0 . \quad ... (6.3)$$

We define the following quantities:

$$H_0 = \frac{p^2}{2\mu} - \frac{Ze^2}{r}$$
$$H_B = H_0 + \frac{\Lambda}{2\mu r^2} \quad ... (6.4)$$

where the subscript B stands for binomial. The consistency conditions for this system are:

$$\left[\mathrm{L}, H_g\right] \approx 0$$
$$\left[\hat{k}, H_g\right] \approx 0 . \quad ... (6.5)$$

First we must derive the form of the unit Runge vector in this problem. It is derived in Appendix A. We arrive at the result:

$$\hat{k} = \frac{1 + g\,g_0}{\sqrt{1 + g^2 + g_0^2 + g^2 g_0^2}}\hat{r} - \frac{\mathbf{p}\cdot\mathbf{r}}{Lr}\left(\frac{g_0 f}{1 + g\,g_0} - \frac{g\left(1 + g_0^2\right)}{(1 + g\,g_0)^2}f^3\right)\left(\frac{g - g^3}{u - u_3}\right)\hat{L}\times\hat{r} . \quad ... (6.6)$$

where

$$f = \frac{1 + g\,g_0}{\sqrt{1 + g^2 + g_0^2 + g^2 g_0^2}} . \quad ... (6.7)$$

and

$$\frac{\partial f}{\partial u} = \left[\frac{g_0}{\sqrt{1+g^2+g_0^2+g^2 g_0^2}} - \frac{(1+g\,g_0)(g+g\,g_0^2)}{(1+g^2+g_0^2+g^2 g_0^2)^{3/2}} \right] \left(\frac{\partial g}{\partial u} \right), \quad \text{... (6.8)}$$

The functions g and are g_0 defined in Appendix A. The unit Runge vector as appears in Eq. (6.6) is a general form for any value of the parameter Λ. Hereafter, however, we only consider a small perturbation in the binomial potential such that $\Lambda \ll L^2$. We therefore perform a Taylor series expansion about $\Lambda = 0$ and keep only terms linear in Λ:

$$\hat{k} = \hat{k}^{(\Lambda=0)} + \Lambda \left. \frac{d\hat{k}}{d\Lambda} \right|_{\Lambda=0} \quad \text{... (6.9)}$$

where $\hat{k}^{(\Lambda=0)}$ denotes the unperturbed unit Runge vector, which, by definition, is equal to the normalized classical Laplace-Runge-Lenz vector. The derivative term is fully worked out in Appendix A.

The next step is to calculate the Poisson brackets given in Eq. (6.5) to arrive at a functional form of the consistency conditions and thus solve for the unknown vector coefficients **u** and **w**. We begin with the angular momentum bracket:

$$\left[L_i, H_g\right] = \left[L_i, H_0\right] + \left[L_i, \frac{\Lambda}{2\mu r^2}\right] + u_j \left[L_i, \left(L_j - L_{0_j}\right)\right] + w_j \left[L_i, \left(\hat{k}_j - \hat{k}_{0_j}\right)\right] \quad \text{... (6.10)}$$

Clearly the first two terms vanish since the angular momentum is conserved in any centrally symmetric potential in the absence of external forces. So we obtain:

$$\left[\mathrm{L}, H_g\right] = \mathbf{u} \times \mathbf{L} + \mathbf{w} \times \hat{k} \approx 0 \quad \text{... (6.11)}$$

We now proceed to the calculation of the time derivative of the unit Runge vector via the Poisson bracket:

$$\left[\hat{k}, H_g\right] = \left[\hat{k}_i, H_0\right] + \left[\hat{k}_i, \frac{\Lambda}{2\mu r^2}\right] + u_j \left[\hat{k}_i, \left(L_j - L_{0_i}\right)\right] + w_j \left[\hat{k}_i, \left(\hat{k}_j - \hat{k}_{0_j}\right)\right] \approx 0. \quad \text{... (6.12)}$$

The following result is obtained:

$$\left[\hat{k}, H_g\right] = \mathbf{u} \times \hat{k} \approx 0 \quad \text{... (6.13)}$$

A well-known relation between A, L, and H_0 is:

$$A = \sqrt{1 + \frac{2H_0 L^2}{\mu Z e^2}} \quad \text{... (6.14)}$$

We seek the unknown vector coefficients in the following form:

$$\mathbf{u} = a_1 \hat{k}_0 + a_2 \mathbf{L}_0 + a_3 \hat{k}_0 \times \mathbf{L}_0$$

$$\mathbf{w} = b_1 \hat{k}_0 + b_2 \mathbf{L}_0 + b_3 \hat{k}_0 \times \mathbf{L}_0 \quad \text{... (6.15)}$$

Substituting Eq. (6.15) into Eq. (6.11) yields:

$$a_1 \hat{k}_0 \times \mathbf{L}_0 + a_3 \left(\hat{k}_0 \times \mathbf{L}_0\right) \times \mathbf{L}_0 - b_2 \hat{k}_0 \times \mathbf{L}_0 + b_3 \left(\hat{k}_0 \times \mathbf{L}_0\right) \times \hat{k}_0 \approx 0. \quad \text{... (6.16)}$$

For this expression to vanish and from Eq. (6.31) we conclude that

$$a_1 = b_2 \quad \text{and} \quad a_2 = a_3 = b_3 = 0. \quad \text{... (6.17)}$$

We now have:

$$\mathbf{u} = a_1 \hat{k}_0$$

$$\mathbf{w} = b_1 \hat{k}_0 + a_1 \mathbf{L}_0. \quad \text{... (6.18)}$$

Now we need to find a_1 and b_1 in terms of the coordinates, momenta and integrals of the motion. Oks and Uzer [4] achieved this for the Coulomb potential by calculating—at the similar stage—the equations of motion of **r** and **p**. However, we found that in our case, the use of the unit Runge vector makes these calculations very tedious (see Appendix C, where we calculated the equations of the motion).

Luckily, an alternative is available. Instead, the coefficients sought may be found in a much simpler and straightforward manner by calculating the frequency of precession of the Runge-Lenz vector (which, by definition, is equal to the precession frequency of the unit Runge vector). The derivation of the frequency of precession of the Laplace-Runge-Lenz vector and further details on finding the unknown vector coefficients are given in Appendix B.

We remind that for any vector **D** that precesses in the plane of motion, the following equation holds

$$\frac{d\mathbf{D}}{dt} = \boldsymbol{\omega}_{precession} \times \mathbf{D} \Rightarrow \omega_{precession} = \frac{1}{D}\frac{dD}{dt} \quad \text{... (6.19)}$$

since the frequency vector is perpendicular to the plane of the orbit. Also, it should be noted that in accordance to the second part of Eq. (6.19) we need to deal only with absolute values of vector **D**.

The precession of the Kepler orbit, caused by the additional term in the binary Hamiltonian (by the term additional to the Coulomb potential), leads—generally speaking—to oscillations of both the eccentricity of the orbit

$$\varepsilon = \frac{A}{Ze^2} \quad \text{... (6.20)}$$

and of the square absolute value of the Laplace-Runge-Lenz vector

$$A^2 = 1 + \frac{2H_0L^2}{\mu Ze^2} = 1 + \frac{2L^2}{\mu Ze^2}\left(H_B - \frac{\Lambda}{2\mu r^2}\right) \quad \text{... (6.21)}$$

Following Oks and Uzer [4], we will investigate the case of *radiationless states*, i.e. states in which the classically-calculated radiation of the electron orbiting the nucleus vanishes. In accordance to Oks and Uzer [4], in the radiationless states the electron has zero velocity, but a non-zero momentum—this became possible due to inclusion of the integral of the motion as constrains into the generalized Hamiltonian. Therefore, in the radiationless states the oscillations of the absolute value of the Laplace-Runge-Lenz vector should also vanish:

$$\frac{dA^2}{dt} = 0. \quad \text{... (6.22)}$$

This condition will allow finding the unknown coefficients in (6.18) in terms of constants of the motion.

The calculation of the left side of Eq. (6.22) is, of course, carried out through the Poisson bracket formalism and is given by the following expression:

$$\frac{dA^2}{dt} = \left[A^2, H_g\right] = \frac{2L^2}{\mu Ze^2}\left[\left(H_B - \frac{\Lambda}{2\mu r^2}\right), H_g\right]$$

$$= -\frac{\Lambda L^2}{\mu^2 Ze^2}\left[\frac{1}{r^2}, H_g\right] \quad \text{... (6.23)}$$

where the first Poisson bracket is of the binomial Hamiltonian with the generalized Hamiltonian and must vanish since the binary potential is conservative, and the bracket containing the square of the angular momentum must necessarily vanish in a central potential. Since we are concerned only with the first order contributions in terms of Λ, then in the right side of Eq. (6.23) it is sufficient to calculate all factors next to Λ in the zeroth order. The details of the calculations of the right side of Eq. (6.23) are presented in Appendix B. The result obtained for the frequency, hereafter the generalized frequency ω_g, is:

$$\omega_g = \frac{\Lambda L^2}{\mu^2 Ze^2\left(1 + \frac{2H_0L_0}{\mu Ze^2}\right)}\mathbf{r}\cdot\mathbf{p}$$

$$\left(\left\langle\frac{1}{r^4}\right\rangle + \frac{b_1}{\left(1 + \frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle} - \frac{L_0^2}{\mu Ze^2\sqrt{1 + \frac{2H_0L_0}{\mu Ze^2}}}\right)\right)$$

$$= \frac{\Lambda L^2}{\mu^2 Ze^2\left(1 + \frac{2H_0L_0}{\mu Ze^2}\right)}\left\langle\frac{1}{r^4}\right\rangle\mathbf{r}\cdot\mathbf{p}$$

$$\left(1 + \frac{b_1}{\left(1 + \frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle} - \frac{L_0^2}{\mu Ze^2\sqrt{1 + \frac{2H_0L_0}{\mu Ze^2}}}\right)\right)$$

$$\equiv \omega_0\left|1 + B\left(H_0, L_0\right)\right|. \quad \text{... (6.24)}$$

The above expression for the generalized frequency contains only one coefficient b_1 that still has to be determined.

One of the central points is that the generalized frequency varies from the classical frequency by a factor denoted in Eq. (6.24) as $\left|1 + B\left(H_0, L_0\right)\right|$. This is equivalent to the following time transformation:

$$t \to t' = t/\left(1 + B\left(H_0, L_0\right)\right) \quad \text{... (6.25)}$$

This is a *new kind of non-Einsteinian time dilation*. The radiationless states correspond to $B\ (H_0, L_0) = -1$. Indeed, when $B\ (H_0, L_0)$ reaches – 1, the *time becomes dilated by the infinite factor and the classical radiation vanishes*.

Upon substitution of this scaled time into all calculations, all quantities regain their standard functional form.

This is in complete agreement with the results of Oks and Uzer in [4]. With Eq. (6.25) in mind, we note that the generalized period of the motion of the electron about the nucleus is

$$T_g = \frac{T}{\left|1+B\left(H_0, L_0\right)\right|}. \qquad \ldots (6.26)$$

At this point it is necessary to point out that the equations relevant to the results derived for the generalized frequency resulted independent of a_1 and, therefore, without loss of generality we may set $a_1 = 0$ in the generalized Hamiltonian. Here, as in Appendix B, we substitute Eq. (B.19) into the generalized Hamiltonian and obtain:

$$H_g = H_B + \frac{\mu Z e^2 B\left(H_0, L_0\right)\left(1+\frac{2H_0L_0}{\mu Z e^2}\right)}{8L^2\left(\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{\sqrt{1+\frac{2H_0L_0}{\mu Z e^2}}}\right)}\left(\hat{k}\cdot\hat{k}_0-1\right)$$

$$= H_B + \frac{\mu Z e^2 B\left(H_0, L_0\right) A^2}{8L^2\left(\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{A}\right)}\left(\hat{k}\cdot\hat{k}_0-1\right). \qquad \ldots (6.27)$$

With the goal of a generalized frequency for non-radiating states of motion of the electron, we note that following Oks and Uzer [4], the generalized frequency from (6.24), may be rewritten as:

$$\omega_g = \omega_0\left|1+B\left(H_0, L_0\right)\right| = \left(\frac{\left|2H_0\right|^3}{\mu Z e^2}\right)^{\frac{1}{2}}\left|1+B\left(H_0, L_0\right)\right| \qquad \ldots (6.28)$$

where, again, the quantities differ from their standard value by the factor $1 + B$.

Equation (6.28) illustrates once again one of the central results of the GHD: the generalized frequency can vanish (i.e. $\omega_g = 0$) despite a non-zero standard classical frequency $\omega_0 \neq 0$. This occurs when $B(H_0, L_0) = -1$, corresponding to a stable, nonradiating state of the classical atom.

For determining the last unknown coefficient b_1—and thus $B(H_0, L_0)$—we employ the same experimental fact as used by Oks and Uzer [4]. Namely, highly excited atoms primarily emit radiation at a non-zero, finite frequency determined by the limit $H_0 \to 0$. Thus, it is expected that there exists a limiting value for the generalized frequency as the Coulomb Hamiltonian approaches zero. We have

$$\lim_{H_0\to 0}\omega_g = \lim_{H_0\to 0}\left(\frac{\left|2H_0\right|^3}{\mu Z e^2}\right)^{\frac{1}{2}}\left|1+B\left(H_0, L_0\right)\right| \equiv \Omega \qquad \ldots (6.29)$$

and this yields:

$$B\left(H_0, L_0\right) = \Omega\left(\frac{\mu Z e^2}{\left|2H_0\right|^3}\right)^{\frac{1}{2}} - 1 \approx \Omega\left(\frac{\mu Z e^2}{\left|2H_0\right|^3}\right)^{\frac{1}{2}} = \frac{\Omega}{\omega_0} \qquad \ldots (6.30)$$

the contribution of -1 is negligible since in the limit as H_0 approaches zero, the term containing Ω predominates.

Now let us consider a stable, radiationless state, so that $\omega_g = 0$. In this stable state we denote $H_0 = H_S$ (the subscript S is for "stable"). We must have:

$$B\left(H_S, L_s\right) = -1, \qquad \ldots (6.31)$$

thus

$$\Omega = -\left|2H_S\right|^{\frac{3}{2}}\left|\mu Z e^2\right|^{-\frac{1}{2}} = -\sqrt{\frac{8}{\mu Z e^2}}\left|H_S\right|^{\frac{3}{2}}$$

$$= -\omega_0\left(H_S, L_S\right) \qquad \ldots (6.32)$$

and

$$B\left(H_0, L_0\right) = -\left|\frac{H_S}{H_0}\right|^{\frac{3}{2}} = -\frac{\omega_0\left(H_S, L_S\right)}{\omega_0\left(H_0, L_0\right)}. \qquad \ldots (6.33)$$

Upon substitution of Eq. (6.32) and (6.33) into Eq. (6.28) we arrive at:

$$\omega_g = \omega_0\left(H_S, L_S\right) - \omega_0\left(H_0, L_0\right). \qquad \ldots (6.34)$$

We find then that the average frequency in the classical process of radiation in a weakly bound state is given by:

$$\left\langle\omega_g\right\rangle \approx \frac{\omega_g^{initial}+\omega_g^{final}}{2} \approx \frac{\omega_g^{initial}}{2}$$

$$= \frac{\omega_0\left(H_S, L_S\right)-\omega_0\left(H_0, L_0\right)}{2} \approx \frac{\omega_0\left(H_S, L_S\right)}{2} \qquad \ldots (6.35)$$

where the final frequency is taken to vanish since there should no longer be any radiation in the final state of motion

and we also used the fact that $\omega_0(H_S, L_S) >> \omega_0(H_0, L_0)$.

At this point, in keeping with the treatment of the problem as in [4], we take into consideration Planck's hypothesis. According to Plank, the smallest possible change in energy is proportional to the frequency of the motion, and the proportionality constant is the Planck's constant in SI units. In our particular problem, however, this is not so simple because, as is established in Holas and March [6], the unit Runge vector is only piecewise continuous, reflecting the well-known fact that the motion in the modified Coulomb potential is only conditionally periodic (as opposed to periodic). For a conditionally-periodic motion in a central field, the relation between changes of the energy and of the angular momentum should be refined as follows:

$$\int_0^{T_r} \Delta E dt = \oint \Delta L d\theta = \int_0^{T_\theta} \omega \Delta L dt \qquad \text{... (6.36)}$$

where T_r is the period of radial motion and T_θ is the period angular motion. Equation (6.36) is justified by the fact the change in energy, $\Delta E/\Delta t = \omega \Delta M/\Delta t$, is commesurate with the change, in this case a decrease, of the size of the orbit. Therefore, the integral in the left side of Eq. (6.36), containing the energy, should be over the period of radial motion. In the right side of Eq. (6.36), the integral contains the angular momentum, which is the variable canonically conjugate to the angular variable θ, therefore the integration should be performed over the period of angular motion.

Combining Eq. (6.36) with Planck's hypothesis we get:

$$\int_0^{T_r} \Delta E dt = \oint \Delta L d\theta = \int_0^{T_\theta} \omega \Delta L dt \Rightarrow \frac{T_r}{T_\theta} \Delta E$$

$$= \langle \omega \Delta L \rangle = \hbar \langle \omega \rangle \qquad \text{... (6.37)}$$

In Eq. (6.37), the change in energy must, of course, satisfy the relation

$$\Delta E = |H_S| - |H_0| \approx |H_S|$$

$$= \hbar \langle \omega_g \rangle = \frac{\hbar}{2} \omega_0 (H_S, L_S) \qquad \text{... (6.38)}$$

or

$$|H_S| \approx \frac{\hbar}{2} \omega_0 (H_S, L_S). \qquad \text{... (6.39)}$$

We note that in both sides of the Eq. (6.39) only physical quantities pertaining to the stable states are present. Also, in Eq. (6.37) we have

$$\frac{T_r}{T_\theta} = \frac{\omega_\theta}{\omega_r} = \frac{1}{\gamma}; \quad \gamma = \sqrt{1 + \frac{\Lambda}{L^2}} \qquad \text{... (6.40)}$$

We note that as $\Lambda \to 0$, $\gamma = \sqrt{1 + \frac{\Lambda}{L^2}} \to 1$, which implies that $T_r = T_\theta$, as known for the Coulomb potential. Thus we have

$$\Delta E = \frac{1}{\gamma} |H_S| \approx \frac{\hbar}{2} \omega_0 (H_S, L_S)$$

$$= \frac{\hbar}{2} \frac{(n\omega_r + m\omega_\theta)}{2} = \frac{\hbar}{2} \left(n + \frac{m}{\gamma} \right) \frac{\omega_r}{2}$$

$$= \hbar \left(n + \frac{m}{\gamma} \right) \sqrt{\frac{1}{2\mu Z^2 e^4}} |H_S|^{\frac{3}{2}} \qquad \text{... (6.41)}$$

where $n, m = 1, 2, 3\ldots$. In the third step of Eq. (6.41) we used the relation between the frequencies given in Eq. (6.40) and we substituted $\omega_0(H_S, L_S)$ for the term $\frac{(n\omega_r + m\omega_\theta)}{2}$, which is the average of the two frequencies throughout the motion (hence the 1/2); and, further, the expression must be valid not only for the first harmonic, but for all occurring harmonics of the radial and angular frequencies, hence the integer factors *n* and *m*. We have also used:

$$\omega_{0\gamma} = \frac{\omega_{0r}}{\gamma} = \sqrt{\frac{8}{\mu Z^2 e^4}} \frac{|H_S|^{\frac{3}{2}}}{\sqrt{1 + \frac{2\mu Z e}{L^2}}}. \qquad \text{... (6.42)}$$

Equation (6.41) shows, in particular, that

$$\frac{1}{\gamma} |H_S| = \hbar \left(n + \frac{m}{\gamma} \right) \sqrt{\frac{1}{2\mu Z^2 e^4}} |H_S|^{\frac{3}{2}} \qquad \text{... (6.43)}$$

Solving Eq. (6.43) for H_S, we obtain the final the expression for the absolute value of *energy of classical non-radiating states* in terms of the integers *n* and *m*:

$$|H_S| = \frac{2\mu Z^2 e^4}{\hbar^2 (n\gamma + m)^2}; \quad n, m = 1, 2\ldots \qquad \text{... (6.44)}$$

Now we compare this classically-derived result with the known quantal result, which can be found, e.g. in the textbook [18] in problem 3 after Section 36:

$$\left|H_{quantal}\right| = \frac{2\mu Z^2 e^4}{\hbar^2\left((2\ell+1)\gamma+(2n_r+1)\right)^2}; n_r, \ell = 0, 1, 2... \quad ... (6.45)$$

In Eq. (6.45), n_r and ℓ are the radial and angular momentum quantum numbers, respectively. We see that in the quantal result, the ground state (ℓ, n_r = 0), agrees exactly with our classical expression (6.44) for n, m = 1. Furthermore, our classical result coincides with the quantal result for all odd n and m, i.e. when these integers are of the form $n = 2k + 1$ and $m = 2q + 1$, where q, k = 0, 1, 2….

We believe that the expression $t \to t' = t/\left(1+B\left(H_0, L_0\right)\right)$ from Eq. (6.25), representing the new phenomenon of the non-Einsteinian time dilation, is valid for arbitrary Λ rather than only for $\Lambda << 1$. This is because the expression (6.44) for the energy of classical stable, non-radiating states is in a good agreement with the corresponding quantal result, the latter being valid for arbitrary Λ.

7. CONCLUSIONS

We applied Dirac's Generalized Hamiltonian Dynamics (GHD) to studying so-called binomial potential, i.e. the Coulomb potential plus an additional term proportional to $1/r^2$. We obtained an explicit expression for the additional (to the angular momentum) vector integral of the motion for the binomial potential: the unit Runge-Lenz vector. In spirit of the GHD, we used the unit Runge-Lenz vector and the angular momentum vector as constraints added to the classical Hamilton function (called Hamiltonian for brevity). Using consistency conditions, we derived the explicit expression for the generalized classical Hamiltonian and showed that it leads to a much richer dynamics than the usual classical dynamics for the same potential.

Then following the logic from Oks-Uzer's paper [4], we obtained classical stable, non-radiating states — classical "discrete" states — for a spinless charged particle, such as, e.g. pion (or kaon) in the binomial potential. We showed that energies of these classical discrete states agree with the corresponding quantal results for pionic atoms in the ground state and in all states of odd principal and angular momentum quantum numbers. We demonstrated that these results can be interpreted as a new (non-Einsteinian) time dilation, like in Oks-Uzer's paper [4].

It is worth emphasizing some interesting physics of the classical stable, non-radiating states following paper [4]. In those states, $\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{p}}{dt} = 0$, so that $\mathbf{r}(t) = \mathbf{r}_0$ and $\mathbf{p}(t) = \mathbf{p}_0$, where $\mathbf{r}_0$ and $\mathbf{p}_0$ are some constant vectors. Thus, the particle (for example, the pion) is motionless, but its momentum is nonzero. This is not surprising: for example, for a charge in an electromagnetic field characterized by a vector potential $\mathbf{A}$, it is also possible to have $\mathbf{v} = \frac{\mathbf{p} - \frac{e}{mc}\mathbf{A}}{m} = 0$, while $\mathbf{p} = \frac{e}{mc}\mathbf{A} \neq 0$.

Another interpretation — complementing and consistent with the above one — is the following. Let us consider a pionic atom in an arbitrary classical state (not in one of the stable states). Due to a non-zero acceleration of the pion, the atom radiates and its energy diminishes. As its energy approaches the nearest classical stable state, the time gets more and more dilated. As the atom reaches the latter state, the time stops and so does the radiation.

There is nothing miraculous in the fact that stable, non-radiating states of atomic and molecular systems can be obtained classically via the GHD. Indeed, let us point out that formal mathematical solutions of the Schrödinger, Klein-Gordon, and Dirac equations do not show by themselves any "quantization", any discreteness property. Discrete energy levels (quantization) are obtained by imposing *constraints* on the formal solutions — constraints in the form of boundary conditions (including conditions at the origin and at infinity). Having this in mind, it might be now less surprising that the inclusion of *constraints* in the classical Hamiltonian also leads to stable states characterized by a discrete set of energies.

In our study we ended up with twice as many stable states as in the corresponding quantal problem. Namely, the classical stable states, characterized be even harmonic numbers n and m in Eq. (6.44), do not have quantal counterparts. It would be interesting to try obtaining the corresponding experimental results for pionic (or kaonic) atoms and to compare them with the predictions of the two theories. However, it should be emphasized that for pionic (or kaonic) atoms of a relatively small nuclear charge, the difference relates only to the fine structure

within the multiplet of a given principal quantum number N—without affecting the primary energy scaling, where $|E|$ is proportional to $1/N^2$.

Regardless of the possible outcome of such benchmark experiments (which are difficult to conduct), we would like to make the following final comments. Any classical formalism has obvious advantages over quantal formalisms—because quantal formalisms deal with operators and therefore lack "transparency" and an intuitive perception. Nevertheless, we do not advocate that the classical GHD is better than the quantal formalism. Rather we insist on *pluralism* in the analytical foundations of atomic and molecular phenomena. One of our intentions is to illuminate the variety of the underlying theories and to provide a stimulus for a better physical understanding of atomic and molecular phenomena.

After all, any physical theory is not the ultimate truth, but just a model, whose limitations—if not known now—will be discovered in the future. Let us refer to the example that for the overwhelming majority of practical purposes, the 4 D-space-time "slice" of 10 D string theory, or of the 11 D M-theory, or of the 12D F-theory (see, e.g., [19]), is a sufficient useful model. Similarly, we believe that despite being just a model, the applications of the classical GHD to atomic and molecular phenomena are useful and will be further developed.

APPENDIX A
Derivation of the Explicit Form of the Unit Runge-Lenz Vector for the Binomial Potential

The function f, given by

$$f = \cos\theta;\theta = \frac{L}{\mu}\int_{u_0}^{u}\left[\frac{2}{\mu}\left(E - V\left(1/u'\right)\right) - \left(\frac{L_{eff}u'}{\mu}\right)^2\right]^{-1/2} du' \quad \text{... (A.1)}$$

where $V(r)$ is the Coulomb part of the binomial potential and

$$L_{eff}^2 = L^2 - \Lambda \quad \text{... (A.2)}$$

is the effective angular momentum. The second term in (A.2) corresponds to the presence of the term proportional to $1/r^2$ in the binomial potential. The integral in Eq. (A.1), upon the substitution of the Coulomb potential, may be rewritten as:

$$\int_{u_0}^{u}\left[\frac{2}{\mu}\left(E - V\left(1/u'\right)\right) - \left(\frac{L_{eff}u'}{\mu}\right)^2\right]^{-1/2} du'$$

$$= \int_{u_0}^{u}\frac{du'}{\sqrt{\left(\frac{2}{\mu}\right)\left(E + Ze^2u'\right) - \left(\frac{Lu'}{\mu}\right)^2}}. \quad \text{... (A.3)}$$

If we now introduce the substitutions

$$u_1 = \frac{Ze^2\mu}{L^2}\left(1 + \sqrt{\frac{2EL^2}{Z^2e^4\mu}}\right), \quad \text{... (A.4)}$$

then the left-hand side of Eq. (A.3), in the indefinite form of the integral, becomes:

$$\int \frac{du'}{\sqrt{(u_1 - u)(u - u_2)}}$$

$$= \tan^{-1}\left(\frac{u - \frac{u_1 + u_2}{2}}{\sqrt{(u_1 - u)(u - u_2)}}\right) \quad \text{... (A.5)}$$

after some simplifications. It is convenient to define

$$u_3 = \frac{u_1 + u_2}{2} = \frac{\mu Ze^2}{L^2} \quad \text{... (A.6)}$$

and thus Eq. (A.5) reduces to:

$$\tan^{-1}\left(\frac{u - \frac{u_1 + u_2}{2}}{\sqrt{(u_1 - u)(u - u_2)}}\right)$$

$$= \tan^{-1}\left(\frac{u - u_3}{\sqrt{(u_1 - u)(u - u_2)}}\right). \quad \text{... (A.7)}$$

Putting in the limits of integration yields:

$$f = \cos\left[\begin{array}{l}\tan^{-1}\left(\dfrac{u-u_3}{\sqrt{(u_1-u)(u-u_2)}}\right)\\ -\tan^{-1}\left(\dfrac{u_0-u_3}{\sqrt{(u_1-u_0)(u_0-u_2)}}\right)\end{array}\right]. \quad \text{... (A.8)}$$

It is convenient to define:

$$g = g(u) \equiv \frac{u-u_3}{\sqrt{(u_1-u)(u-u_2)}};\ g(u_0) \equiv g_0. \quad \text{... (A.9)}$$

Using the identity

$$\cos\left(\tan^{-1}(A) - \tan^{-1}(B)\right) = \frac{1+AB}{\sqrt{1+A^2+B^2+A^2B^2}}, \quad \text{... (A.10)}$$

we may then write

$$f = \frac{1+g\,g_0}{\sqrt{1+g^2+g_0^2+g^2g_0^2}}. \quad \text{... (A.11)}$$

Consequently, the partial derivative in the unit Runge vector becomes:

$$\frac{\partial f}{\partial u} = \left[\begin{array}{l}\dfrac{g_0}{\sqrt{1+g^2+g_0^2+g^2g_0^2}}\\ -\dfrac{(1+g\,g_0)(g+g\,g_0^2)}{(1+g^2+g_0^2+g^2g_0^2)^{3/2}}\end{array}\right]\left(\frac{\partial g}{\partial u}\right), \quad \text{... (A.12)}$$

where

$$\frac{\partial g}{\partial u} = \frac{1}{\sqrt{(u_1-u)(u-u_2)}} - \frac{1}{2}\frac{\left(u-\frac{u_1+u_2}{2}\right)(-2u+u_1+u_2)}{((u_1-u)(u-u_2))^{3/2}}. \quad \text{... (A.13)}$$

We may use the definitions (A.9) and (A.11) to rewrite Eq. (A.13) and put it into Eq. (A.12) to get the following compact form:

$$\frac{\partial f}{\partial u} = \left(\frac{g_0 f}{1+g\,g_0} - \frac{g\left(1+g_0^2\right)}{(1+g\,g_0)^2}f^3\right)\left(\frac{g-g^3}{u-u_3}\right). \quad \text{... (A.14)}$$

where the term in the second set of parenthesis is the simplification of $\frac{\partial g}{\partial u}$. We thus arrive at:

$$\hat{k} = \frac{1+g\,g_0}{\sqrt{1+g^2+g_0^2+g^2g_0^2}}\hat{r} - \frac{\mathbf{p}\cdot\mathbf{r}}{Lr}\left(\frac{g_0 f}{1+g\,g_0} - \frac{g\left(1+g_0^2\right)}{(1+g\,g_0)^2}f^3\right)\left(\frac{g-g^3}{u-u_3}\right)\hat{L}\times\hat{r}. \quad \text{... (A.15)}$$

This is a general result valid for any value of Λ. However, since we are considering a small perturbation in the binomial potential, such that $\Lambda \ll L_2$, then we may perform a Taylor series expansion of the unit Runge vector with respect to Λ about $\Lambda = 0$:

$$\hat{k} = \hat{k}^{(\Lambda=0)} + \Lambda\left.\frac{d\hat{k}}{d\Lambda}\right|_{\Lambda=0} \quad \text{... (A.16)}$$

where $\hat{k}^{(\Lambda=0)}$ denotes the unperturbed unit Runge vector, which, by definition, is equal to the normalized classical Laplace-Runge-Lenz vector. Differentiation with respect to Λ yields:

$$\frac{d\hat{k}}{d\Lambda} = \frac{df}{d\Lambda}\hat{r} - \frac{\mathbf{p}\cdot\mathbf{r}}{L_{eff}\,r}\left(\frac{d}{d\Lambda}\left(\frac{\partial f}{\partial u}\right) + \frac{1}{2L_{eff}^2}\frac{\partial f}{\partial u}\right)\hat{L}\times\hat{r}. \quad \text{... (A.17)}$$

The second term in the parenthesis is due to $\frac{\partial L_{eff}}{\partial \Lambda} = -\frac{1}{2L_{eff}}$. We now proceed to calculate the above quantities.

For the first term:

$$\frac{df}{d\Lambda} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial \Lambda} + \frac{\partial f}{\partial g_0}\frac{\partial g_0}{\partial \Lambda} \quad \text{... (A.18)}$$

where

$$\frac{\partial f}{\partial g} = \frac{g_0 f}{1+g\,g_0} - \frac{g\left(1+g_0^2\right)}{(1+g\,g_0)^2}f^3;$$

$$\frac{\partial f}{\partial g_0} = \frac{gf}{1+g\,g_0} - \frac{g_0\left(1+g^2\right)}{\left(1+g\,g_0\right)^2} f^3 \quad \text{... (A.19)}$$

as can be seen in Eq. (A.14); and

$$\frac{\partial g}{\partial \Lambda} = -\frac{1}{2L_{eff}} \frac{\partial g}{\partial u_i} \frac{\partial u_i}{\partial L_{eff}};$$

$$\frac{\partial g_0}{\partial \Lambda} = -\frac{1}{2L_{eff}} \frac{\partial g_0}{\partial u_i} \frac{\partial u_i}{\partial L_{eff}} \quad i = 1, 2, 3 \quad \text{... (A.20)}$$

We find:

$$\frac{\partial g}{\partial u_1} = -\frac{1}{2} \frac{(u-u_2)}{(u-u_3)^2} g_0^3;$$

$$\frac{\partial u_1}{\partial L_{eff}} = -\frac{2u_1}{L_{eff}} + \frac{\mu}{L_{eff}^2} \sqrt{2\mu E}$$

$$\frac{\partial g_0}{\partial u_1} = -\frac{1}{2} \frac{(u_1-u)}{(u-u_3)^2} g_0^3;$$

$$\frac{\partial u_2}{\partial L_{eff}} = -\frac{2u_2}{L_{eff}} - \frac{\mu}{L_{eff}^2} \sqrt{2\mu E}$$

$$\frac{\partial g_0}{\partial u_3} = -\frac{g_0}{u-u_3};$$

$$\frac{\partial u_3}{\partial L_{eff}} = -\frac{2u_3}{L_{eff}}. \quad \text{... (A.21)}$$

and

$$\frac{\partial g_0}{\partial u_1} = -\frac{1}{2} \frac{(u-u_2)}{(u-u_3)^2} g_0^3;$$

$$\frac{\partial u_1}{\partial L_{eff}} = -\frac{2u_1}{L_{eff}} + \frac{\mu}{L_{eff}^2} \sqrt{2\mu E}$$

$$\frac{\partial g_0}{\partial u_1} = -\frac{1}{2} \frac{(u_1-u)}{(u-u_3)^2} g_0^3;$$

$$\frac{\partial u_2}{\partial L_{eff}} = -\frac{2u_2}{L_{eff}} - \frac{\mu}{L_{eff}^2} \sqrt{2\mu E}$$

$$\frac{\partial g_0}{\partial u_3} = -\frac{g_0}{u-u_3};$$

$$\frac{\partial u_3}{\partial L_{eff}} = -\frac{2u_3}{L_{eff}}. \quad \text{... (A.22)}$$

The second derivative term is found to be:

$$\frac{d}{d\Lambda}\left(\frac{\partial f}{\partial u}\right)$$

$$= \left[\frac{g_0}{1+g\,g_0} - 3\frac{g\left(1+g_0^2\right)}{\left(1+g\,g_0\right)^2} f^2\right] \frac{df}{d\Lambda}\left(\frac{g-g^3}{u-u_3}\right)$$

$$+\left[-\frac{g_0}{\left(1+g\,g_0\right)^2} - \frac{\left(1+g_0^2\right)}{\left(1+g\,g_0\right)^2} f^3 - 2\frac{g_0 g\left(1+g_0^2\right)}{\left(1+g\,g_0\right)^3}\right] \frac{dg}{d\Lambda}\left(\frac{g-g^3}{u-u_3}\right)$$

$$+\left[\frac{g_0 f}{1+g\,g_0} - \frac{g\left(1+g_0^2\right)}{\left(1+g\,g_0\right)^2} f^3\right]\left(\frac{1-3g^2}{u-u_3}\right)\frac{dg}{d\Lambda}$$

$$-\left[\frac{g_0 f}{1+g\,g_0} - \frac{g\left(1+g_0^2\right)}{\left(1+g\,g_0\right)^2} f^3\right]\left[\frac{g-g^3}{(u-u_3)^2}\right]\left(\frac{du}{d\Lambda} - \frac{du_3}{d\Lambda}\right).$$

... (A.23)

We note that substituting $\Lambda = 0$ amounts to the substitution $L_{eff} \to L$ in accordance with Eq. (A.2).

APPENDIX B
Derivation of the Frequency of Precession of the Laplace-Runge-Lenz Vector

We start by explicitly writing the generalized Hamiltonian in Eq. (6.23), substituting the vectors **u** and **w** from Eq. (6.18):

$$H_g = \frac{p^2}{2\mu} - \frac{Ze^2}{r} + \frac{\Lambda}{2\mu r^2}$$

$$+ a_1 \mathbf{L}_0 \cdot \hat{k} + a_1 \mathbf{L} \cdot \hat{k}_0 + b_1 \hat{k} \cdot \hat{k}_0 - b_1 \quad \text{... (B.1)}$$

The Poisson bracket of the binomial term, which is simply a term inversely proportional to the square of the distance, with the generalized Hamiltonian (B.1) is, in accordance with Eq. (6.23):

$$\frac{dA^2}{dt} = -\frac{\Lambda L^2}{\mu^2 Ze^2}\left[\frac{1}{r^2}, H_g\right]$$

$$= -\frac{\Lambda L^2}{\mu^2 Ze^2}\left\{-\frac{2\mathbf{r}\cdot\mathbf{p}}{r^4} + b_1 \hat{A}_0 \left[\frac{1}{r^2}, \frac{\mathbf{A}}{A}\right]\right\}$$

... (B.2)

where we have made the substitution

$$\hat{k}^{(\Lambda=0)} = \frac{\mathbf{A}}{A},$$

which is justified by definition; it has also been taken into account that the Poisson bracket of the binomial potential with the angular momentum vanishes and the bracket

with b_1 must vanish as b_1 is constant. It is worthwhile to recall that the quantities indexed by a 0 are constant and consequently may be factored out of any bracket in which they appear. In (B.2) we note that:

$$\left[\frac{1}{r^2}, \frac{A_q}{A}\right] = \frac{1}{A}\left[\frac{1}{r^2}, A_q\right] + A_q\left[\frac{1}{r^2}, \left(A_j A_j\right)^{-\frac{1}{2}}\right]$$

$$= \frac{1}{A}\left[\frac{1}{r^2}, A_q\right] - \frac{A_q A_j}{A^3}\left[\frac{1}{r^2}, A_j\right] \quad \text{... (B.3)}$$

and

$$\left[\frac{1}{r^2}, A_j\right] = -\frac{2x_i}{r^4}\frac{\partial A_j}{\partial p_i} \quad \text{... (B.4)}$$

since the second term of the bracket is zero because the coordinates and momenta are assumed to be independent of each other. Thus, the entire calculation of the frequency of oscillation of the Runge-Lenz vector has been reduced to the calculation of the derivative of the of the Runge-Lenz vector with respect to the momenta. This is as follows:

$$\frac{\partial A_j}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\frac{1}{\mu Z e^2}\varepsilon_{jkl} p_k L_l - \frac{x_j}{r}\right)$$

$$= \frac{\partial}{\partial p_i}\left(\frac{1}{\mu Z e^2}\varepsilon_{jkl}\varepsilon_{lmn} p_k x_m p_n\right)$$

$$= \frac{1}{\mu Z e^2}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right)$$

$$\left(\delta_{ik} x_m p_n + \delta_{in} x_m p_k\right)$$

$$= \frac{1}{\mu Z e^2}\left(2x_j p_i - x_i p_j - \mathbf{r}\cdot\mathbf{p}\delta_{ij}\right) \quad \text{... (B.5)}$$

so that (B.4) becomes:

$$\left[\frac{1}{r^2}, A_j\right] = -\frac{2}{\mu Z e^2 r^4}\left(x_j \mathbf{r}\cdot\mathbf{p} - p_j r^2\right). \quad \text{... (B.6)}$$

With the simple result found in (B.6), we may now calculate the last part of (B.4) in order to get a final form for (B.3). We have:

$$\frac{1}{A}\left[\frac{1}{r^2}, A_q\right] - \frac{A_q A_j}{A^3}\left[\frac{1}{r^2}, A_j\right]$$

$$= -\frac{2}{\mu Z e^2 r^4 A}\left(x_q \mathbf{r}\cdot\mathbf{p} - p_q r^2\right)$$

$$+ \frac{2}{\mu Z e^2 r^4}\frac{A_q}{A^3}\left(\mathbf{r}\cdot\mathbf{A}\mathbf{r}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{A}r^2\right) \quad \text{... (B.7)}$$

Then, finally, we arrive at:

$$\frac{dA^2}{dt} = \left[A^2, H_g\right] = -\frac{\Lambda L^2}{\mu^2 Z e^2}$$

$$\left(-\frac{2\mathbf{r}\cdot\mathbf{p}}{\mu r^4} + b_1\hat{A}_0\left[\frac{1}{r^2}, \frac{\mathbf{A}}{A}\right]\right)$$

$$= -\frac{2\Lambda L^2}{\mu^2 Z e^2 r^4}\mathbf{r}\cdot\mathbf{p}$$

$$\left(1 + \frac{b_1}{1 + \frac{2H_0 L_0}{\mu Z e^2}}\left(2r - \frac{L_0^2}{\mu Z e^2\sqrt{1 + \frac{2H_0 L_0}{\mu Z e^2}}}\right)\right)$$

... (B.8)

This expression was significantly simplified in form by carrying out the dot product of the coordinates and momenta with the Runge-Lenz vector:

$$\mathbf{r}\cdot\mathbf{A} = \frac{L^2}{\mu Z e^2} - r \; ; \qquad \mathbf{p}\cdot\mathbf{A} = \frac{\mathbf{p}\cdot\mathbf{r}}{r}. \quad \text{... (B.9)}$$

Furthermore, since our calculations are limited to the first order in terms of Λ, then every factor next to Λ in the right side of (B.8) can be calculated in the zeroth order in terms of Λ. Therefore it is legitimate to replace the quantities $1/r^4$ and $1/r^3$ in the right side of (B.8) by their averages over the unperturbed Kepler ellipse.

These averages are determined as follows:

$$\left\langle\frac{1}{r^3}\right\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{p}{1+\varepsilon\cos\theta}\right)^3 d\theta = \frac{2+3\varepsilon^2}{2p^3}$$

... (B.10a)

$$\left\langle\frac{1}{r^4}\right\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{p}{1+\varepsilon\cos\theta}\right)^4 d\theta = \frac{8+24\varepsilon^2+3\varepsilon^4}{8p^4}$$

... (B.10b)

where

$$p = \frac{L^2}{\mu Z e^2}; \quad \text{and} \quad \varepsilon = \frac{A}{Z e^2} \quad \text{... (B.11)}$$

We shall keep the definitions of the momentum and the eccentricity as in (B.11) for the sake of brevity, but it is to be understood that these quantities are in terms of constants of the motion. Now substitution of (B.11a, b) into (B.8) yields:

$$\frac{2\Lambda L^2}{\mu^2 Ze^2}\mathbf{r}\cdot\mathbf{p}$$

$$\left(\left\langle\frac{1}{r^4}\right\rangle+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\left\langle\frac{1}{r^3}\right\rangle-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\left\langle\frac{1}{r^4}\right\rangle\right)\right)$$

$$=\frac{2\Lambda L^2}{\mu^2 Ze^2}\left\langle\frac{1}{r^4}\right\rangle\mathbf{r}\cdot\mathbf{p}$$

$$\left(1+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)\right)$$

$$=0 \qquad \text{... (B.12)}$$

Next we find the frequency of precession. Since we calculated the time derivative of the square of the magnitude of the Runge-Lenz vector rather than the magnitude to the first power, we make a small correction to (B.12) to arrive at our desired result:

$$\frac{dA^2}{dt}=2A\frac{dA}{dt}\Rightarrow\omega_{precession}=\frac{1}{A}\frac{dA}{dt}$$

$$=\frac{1}{2A^2}\frac{dA^2}{dt} \qquad \text{... (B.13)}$$

We therefore arrive at:

$$\omega_{precession}=\frac{2\Lambda L^2}{\mu^2 Ze^2A^2}\left\langle\frac{1}{r^4}\right\rangle\mathbf{r}\cdot\mathbf{p}$$

$$\left(1+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)\right)$$

$$=\frac{2\Lambda L^2}{\mu^2 Ze^2\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left\langle\frac{1}{r^4}\right\rangle\mathbf{r}\cdot\mathbf{p}$$

$$\left(1+\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)\right)$$

... (B.14)

We may now rewrite this as:

$$\omega_{precession}=\frac{\Lambda L^2\mathbf{r}\cdot\mathbf{p}}{\mu^2 Ze^2A^2}\left\langle\frac{1}{r^4}\right\rangle\left(1+B\left(H_0,L_0\right)\right)$$

... (B.15)

where

$$B\left(H_0,L_0\right)=\frac{b_1}{\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(2\frac{\left\langle\frac{1}{r^3}\right\rangle}{\left\langle\frac{1}{r^4}\right\rangle}-\frac{L_0^2}{\mu Ze^2\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)$$

$$=\frac{b_1}{\mu Ze^2\left(1+\frac{2H_0L_0}{\mu Ze^2}\right)}\left(8L^2\frac{2+3\varepsilon^2}{8+24\varepsilon^2+3\varepsilon^4}-\frac{L_0^2}{\sqrt{1+\frac{2H_0L_0}{\mu Ze^2}}}\right)$$

$$\frac{b_1}{\mu Ze^2A^2}\left(8L^2\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{A}\right),$$

... (B.16)

where in the last step we substituted the expression for the eccentricity and the classical Runge-Lenz vector. We see from (B.15) and (B.16) that in for the case $b_1 = 0$, we recover the well-known classical expression. Thus, the appearance of the function $B\ (H_0, L_0)$ of Eq. (B.18) is a result characteristic of the GHD for the central potential as it depends directly on b_1, the one remaining coefficient from the formalism's constant vectors introduced in the generalized Hamiltonian.

Furthermore, we may solve for b_1 in terms of the Coulomb Hamiltonian, the angular momentum and B:

$$b_1 = \frac{\mu Z e^2 B(H_0, L_0)\left(1+\frac{2H_0L_0}{\mu Z e^2}\right)}{8L^2\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{\sqrt{1+\frac{2H_0L_0}{\mu Z e^2}}}}$$

... (B.17)

or

$$b_1 = \frac{\mu Z e^2 B\left(H_0, L_0\right) A^2}{8L^2\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{A}}$$

... (B.18)

in terms of the classical Runge-Lenz vector. We may now substitute this result into the generalized Hamiltonian to obtain:

$$H_g = H_B + a_1\mathbf{L}_0\cdot\hat{k} + a_1\mathbf{L}\cdot\hat{k}_0$$

$$\frac{\mu Z e^2 B(H_0, L_0)\left(1+\frac{2H_0L_0}{\mu Z e^2}\right)}{8L^2\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{\sqrt{1+\frac{2H_0L_0}{\mu Z e^2}}}}\left(\hat{k}\cdot\hat{k}_0-1\right)$$

... (B.19)

Furthermore, since the equations relevant to all results are independent of a_1, we may, without loss of generality, set $a_1 = 0$, and the Hamiltonian reduces to:

$$H_g = H_B + \frac{\mu Z e^2 B\left(H_0, L_0\right)\left(1+\frac{2H_0L_0}{\mu Z e^2}\right)}{8L^2\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{\sqrt{1+\frac{2H_0L_0}{\mu Z e^2}}}}\left(\hat{k}\cdot\hat{k}_0-1\right)$$

$$= H_B + \frac{\mu Z e^2 B\left(H_0, L_0\right) A^2}{8L^2\frac{2Z^4e^8+3Z^2e^4A^2}{8Z^4e^8+24Z^2e^4A^2+3A^4}-\frac{L_0^2}{A}}\left(\hat{k}\cdot\hat{k}_0-1\right)$$

... (B.20)

The last step was obtained from substitution of (B.19), where we used the magnitude of the classical Runge-Lenz vector in terms of the Coulomb Hamiltonian and the angular momentum. Furthermore, it should be noted that the generalized Hamiltonian is expressed solely as a function of conserved quantities.

APPENDIX C
Derivation of the Equations of the Motion

In classical mechanics the equations of motion for any quantity are given by the Poisson bracket of the quantity with the Hamiltonian for the system. In the GHD, this is extended to the generalized Hamiltonian for the system. Thus we have:

$$\dot{\mathrm{r}} = \left[\mathrm{r}, H_g\right] = \left[\mathbf{r}, H_B\right] + u_i\left[\mathbf{r}, L_i - L_{0_i}\right] + w_i\left[\mathbf{r}, \hat{k}_i - \hat{k}_{0_i}\right]$$

$$\dot{\mathrm{p}} = \left[\mathrm{p}, H_g\right] = \left[\mathbf{p}, H_B\right] + u_i\left[\mathbf{p}, L_i - L_{0_i}\right] + w_i\left[\mathbf{p}, \hat{k}_i - \hat{k}_{0_i}\right].$$

... (C.1)

The equations of motion are known for the pure Coulomb potential, so we only have to calculate the contributions due to the second term in the binomial potential. The calculations yield:

$$\left[x_i, H_B\right] = \frac{p_{0_i}}{\mu} + \left(b_1\hat{k}_j + a_1L_j\right)\left[x_i, \hat{k}_j\right]$$

$$= \frac{\mathbf{p}_0}{\mu} + b_1\left(\begin{array}{l}\delta_{iq}\frac{\partial f}{\partial p_q}rf + \frac{\mathbf{p}\cdot\mathbf{r}}{rL}\delta_{iq}\frac{\partial}{\partial p_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)\\ +\frac{L^2-r^2p^2}{rL^3}\left(\frac{\partial f}{\partial u}\right)^2\left[\hat{L}\times\hat{r}\right]_j\end{array}\right)$$

$$+ b_1\Lambda\frac{d}{d\Lambda}\left(\begin{array}{l}\delta_{iq}\frac{\partial f}{\partial p_q}rf + \frac{\mathbf{p}\cdot\mathbf{r}}{rL}\delta_{iq}\frac{\partial}{\partial p_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)\\ +\frac{L^2-r^2p^2}{rL^3}\left(\frac{\partial f}{\partial u}\right)^2\left[\hat{L}\times\hat{r}\right]_j\end{array}\right)$$

Here we have interchanged the order of differentiation since, by assumption, all functions involved are, at minimum, piecewise continuous and differentiable.

For the momentum we have:

$$\left[p_i, H_g\right] = -\frac{Ze^2}{r^2}\frac{x_i}{r} + \left(b_1\hat{k}_j + a_1L_j\right)\left[p_i, \hat{k}_j\right]$$

$$= -\frac{Ze^2}{r^2}\frac{x_i}{r} - \left(b_1\hat{k}_j + a_1L_j\right)$$

$$\left(\begin{array}{l}\delta_{iq}\frac{\partial f}{\partial x_q}\hat{r} - \delta_{ik}\frac{f}{r^3}x_k - \delta_{iq}\frac{\partial}{\partial x_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)\left[\hat{L}\times\hat{r}\right]_j\\ -\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\left(\begin{array}{l}\left(\frac{1}{r}+\frac{6p^2}{L^2}\right)\frac{x_i}{r}\left[\hat{L}\times\hat{r}\right]_j\\ +\frac{1}{rL}\left(\mathbf{p}\cdot\mathbf{r}\delta_{ij} - 2p_jx_i + p_ix_j\right)\end{array}\right)\end{array}\right)$$

$$-\left(b_1\hat{k}_j + a_1 L_j\right)\Lambda\frac{d}{d\Lambda}$$

$$\begin{pmatrix} \delta_{iq}\dfrac{\partial f}{\partial x_q}\hat{r} - \delta_{ik}\dfrac{f}{r^3}x_k - \delta_{iq}\dfrac{\partial}{\partial x_q}\left(\dfrac{\mathbf{p}\cdot\mathbf{r}}{rL}\dfrac{\partial f}{\partial u}\right)\left[\hat{L}\times\hat{r}\right]_j \\ -\dfrac{\mathbf{p}\cdot\mathbf{r}}{rL}\dfrac{\partial f}{\partial u}\begin{pmatrix}\left(\dfrac{1}{r}+\dfrac{6p^2}{L^2}\right)\dfrac{x_i}{r}\left[\hat{L}\times\hat{r}\right]_j \\ +\dfrac{1}{rL}\left(\mathbf{p}\cdot\mathbf{r}\delta_{ij} - 2p_j x_i + p_i x_j\right)\end{pmatrix}\end{pmatrix}$$

... (C.3)

In Eq. (C.3), we note that the dot product with the angular momentum, the coefficient of a_1, vanishes in all terms except for the term with δ_{ij}. Furthermore, it is important to note that the symmetry of the problem dictates that the evolution of the momentum and of the radius-vector should be contained in the plane of motion. Therefore, the only acceptable value for a_1 is zero— otherwise, the momentum and the radius-vector would have contributions perpendicular to the plane of the orbit. In Eqs (C.2) and (C.3) only the derivatives of the vectors have been worked out fully. This is because the derivatives of scalar quantities are best dealt with as follows:

$$\frac{\partial f}{\partial x_q} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_q} + \frac{\partial f}{\partial L}\frac{\partial L}{\partial x_q}$$

$$= -\frac{\partial f}{\partial u}\frac{x_k}{r^3}\delta_{kq} + \frac{6p^2}{L}\frac{\partial f}{\partial L}x_k\delta_{kq}$$

$$\Rightarrow \delta_{iq}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_q} + \frac{\partial f}{\partial L}\frac{\partial L}{\partial x_q}\right)$$

$$= -\frac{1}{r^3}\frac{\partial f}{\partial u}x_i + \frac{6p^2}{L}\frac{\partial f}{\partial L}x_i \qquad \text{... (C.4)}$$

and

$$\frac{\partial}{\partial x_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)$$

$$= \begin{pmatrix}\dfrac{p_k}{rL}\dfrac{\partial f}{\partial u} - \left(\dfrac{x_k}{r^2}+\dfrac{6p^2}{L^2}x_k\right)\dfrac{\mathbf{p}\cdot\mathbf{r}}{rL}\dfrac{\partial f}{\partial u} \\ +\dfrac{\mathbf{p}\cdot\mathbf{r}}{rL}\dfrac{\partial}{\partial x_q}\left(\dfrac{\partial f}{\partial u}\right)\end{pmatrix}\delta_{kq}$$

$$\Rightarrow \delta_{iq}\frac{\partial}{\partial x_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)$$

$$= \begin{pmatrix}\dfrac{p_i}{rL}\dfrac{\partial f}{\partial u} - \left(\dfrac{x_i}{r^2}+\dfrac{6p^2}{L^2}x_i\right)\dfrac{\mathbf{p}\cdot\mathbf{r}}{rL}\dfrac{\partial f}{\partial u} \\ -\dfrac{\mathbf{p}\cdot\mathbf{r}}{rL}\left(\dfrac{x_i}{r^3}\dfrac{\partial^2 f}{\partial u^2}+3\dfrac{\partial f}{\partial u}\dfrac{x_i}{r^2}+\dfrac{6p^2}{L}\dfrac{\partial^2 f}{\partial L\partial u}x_i\right)\end{pmatrix}$$

... (C.5)

where

$$\frac{\partial^2 f}{\partial u^2} = \begin{pmatrix}\dfrac{g_0}{1+g\,g_0}\dfrac{\partial f}{\partial u} - \dfrac{g_0^2 f}{(1+g\,g_0)^2}\dfrac{\partial g}{\partial u} \\ -\dfrac{\left(1+g_0^2\right)}{(1+g\,g_0)^2}f^3\dfrac{\partial g}{\partial u} \\ +2\dfrac{g\left(1+g_0^2\right)}{(1+g\,g_0)^3}f^3\dfrac{\partial g}{\partial u} \\ -3\dfrac{g\left(1+g_0^2\right)}{(1+g\,g_0)^3}f^2\dfrac{\partial f}{\partial u}\end{pmatrix}\left(\frac{g-g^3}{u-u_3}\right)$$

$$= +\left(\frac{g_0 f}{1+g\,g_0} - \frac{g\left(1+g_0^2\right)}{(1+g\,g_0)^2}f^3\right)\left(\frac{1-3g^2}{u-u_3}\frac{\partial g}{\partial u} - \frac{g-g^3}{(u-u_3)^2}\right). \qquad \text{... (C.6)}$$

For the derivative with respect to the momentum, we find:

$$\frac{\partial f}{\partial p_q} = \frac{\partial f}{\partial L}\frac{\partial L}{\partial p}\frac{\partial p}{\partial p_q}$$

$$= \frac{6r^2}{L}\frac{\partial f}{\partial L}p_k\delta_{kq}$$

$$\Rightarrow \delta_{iq}\left(\frac{6r^2}{L}\frac{\partial f}{\partial L}p_k\delta_{kq}\right)$$

$$= \frac{6r^2}{L}\frac{\partial f}{\partial L}p_i\,. \qquad \text{... (C.7)}$$

and similarly

$$\delta_{iq}\frac{\partial}{\partial p_q}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right) = \frac{x_i}{rL}\frac{\partial f}{\partial u} - \frac{6r\mathbf{p}\cdot\mathbf{r}}{L^3}\frac{\partial f}{\partial u}p_i + \frac{6r\mathbf{p}\cdot\mathbf{r}}{L^2}\frac{\partial^2 f}{\partial L\partial u}p_i \quad \text{... (C.8)}$$

After setting a_1 to zero and carrying out the dot products, the equations of motion become:

$$[\mathbf{r}, H_B] = \frac{p_{0_i}}{\mu} + b_1\hat{k}_j\left[x_i, \hat{k}_j\right]$$

$$= \frac{\mathbf{p}_0}{\mu} + b_1\left(\begin{array}{l}\left(\frac{6r^3 f}{L}\frac{\partial f}{\partial L} - \frac{6(\mathbf{p}\cdot\mathbf{r})^2}{L^3}\left(\frac{1}{L}\frac{\partial f}{\partial u} - \frac{\partial^2 f}{\partial L\partial u}\right)\right)\mathbf{p} \\ + \frac{\mathbf{p}\cdot\mathbf{r}}{(rL)^2}\frac{\partial f}{\partial u}\mathbf{r} + \frac{(\mathbf{p}\cdot\mathbf{r})^2}{rL^3}\left(\frac{\partial f}{\partial u}\right)^2\hat{L}\times\hat{r}\end{array}\right)$$

$$+ b_1\Lambda\frac{d}{d\Lambda}\left(\begin{array}{l}\frac{6r^2}{L}\frac{\partial f}{\partial L}rf\mathbf{p} + \frac{\mathbf{p}\cdot\mathbf{r}}{rL}\left(\begin{array}{l}\frac{1}{rL}\frac{\partial f}{\partial u}\mathbf{r} - \frac{6r\mathbf{p}\cdot\mathbf{r}}{L^3}\frac{\partial f}{\partial u}\mathbf{p} \\ + \frac{6r\mathbf{p}\cdot\mathbf{r}}{L^2}\frac{\partial^2 f}{\partial L\partial u}\mathbf{p}\end{array}\right) \\ + \frac{L^2 - r^2p^2}{rL^3}\left(\frac{\partial f}{\partial u}\right)^2\hat{L}\times\hat{r}\end{array}\right) \quad \text{... (C.9)}$$

$$\left[\mathrm{p}, H_g\right] = -\frac{Ze^2}{r^2}\frac{x_i}{r} + b_1\hat{k}_j\left[p_i, \hat{k}_j\right]$$

$$= -\left(\frac{\partial f}{\partial u} + \frac{6rp^2}{L}\frac{\partial f}{\partial L} + 1\right)\frac{f}{r^2}\hat{r}$$

$$-\frac{Ze^2}{r^2}\hat{r} - b_1\left(\begin{array}{l}-\left(\frac{\partial f}{\partial u} + \frac{6rp^2}{L}\frac{\partial f}{\partial L} + 1\right)\frac{f}{r^2}\hat{r} \\ + \left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)^2\left(\begin{array}{l}\frac{1}{rL}\frac{\partial f}{\partial u} - \left(\frac{1}{r^2} + \frac{6p^2}{L^2}\right) \\ - \frac{\mathbf{p}\cdot\mathbf{r}}{L}\left(\frac{1}{r^2}\frac{\partial^2 f}{\partial u^2} + \frac{3}{r}\frac{\partial f}{\partial u} + \frac{6rp^2}{L}\frac{\partial^2 f}{\partial L\partial u}\right)\end{array}\right)\hat{r} \\ + \left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)^2\left(\begin{array}{l}\left(\frac{1}{r} + \frac{6p^2}{L^2}\right)\hat{r} + \\ \frac{1}{rL}\left(\mathbf{p}\cdot\mathbf{r}\hat{k} - 2\left(\mathbf{p}\cdot\mathbf{r}f - \frac{\mathbf{p}\cdot\mathbf{r}L}{r^2p}\frac{\partial f}{\partial u}\right)\hat{r} + rf\right)\mathbf{p}\end{array}\right)\end{array}\right)$$

$$-b_1\Lambda\frac{d}{d\Lambda}\left(\begin{array}{l}-\left(\frac{\partial f}{\partial u} + \frac{6rp^2}{L}\frac{\partial f}{\partial L} + 1\right)\frac{f}{r^2}\hat{r} \\ + \left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)^2\left(\begin{array}{l}\frac{1}{rL}\frac{\partial f}{\partial u}\mathbf{p} - \left(\frac{1}{r^2} + \frac{6p^2}{L^2}\right) \\ - \frac{\mathbf{p}\cdot\mathbf{r}}{L}\left(\frac{1}{r^2}\frac{\partial^2 f}{\partial u^2} + \frac{3}{r}\frac{\partial f}{\partial u} + \frac{6rp^2}{L}\frac{\partial^2 f}{\partial L\partial u}\right)\end{array}\right)\hat{r} \\ + \left(\frac{\mathbf{p}\cdot\mathbf{r}}{rL}\frac{\partial f}{\partial u}\right)^2\left(\begin{array}{l}\left(\frac{1}{r} + \frac{6p^2}{L^2}\right)\hat{r} \\ + \frac{1}{rL}\left(\mathbf{p}\cdot\mathbf{r}\hat{k} - 2\mathbf{p}\cdot\mathbf{r}\left(f - \frac{L}{r^2p}\frac{\partial f}{\partial u}\right)\hat{r} + rf\mathbf{p}\right)\end{array}\right)\end{array}\right) \quad \text{... (C.10)}$$

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