Surface currents on the plasma-vacuum interface in MHD equilibria

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Abstract

The VMEC[1] non-axisymmetric MHD equilibrium code can compute free-boundary equilibria[2]. Since VMEC assumes that magnetic fields within the plasma form closed and nested flux surfaces, the plasma-vacuum interface is a flux surface, and the total magnetic field there has no normal component. VMEC imposes this condition of zero normal field using the potential formulation of Merkel[3], and solves a Neumann problem for the magnetic potential in the exterior region.

This boundary condition necessarily admits the possibility of a surface current on the plasma-vacuum interface. While this current may be small in MHD equilibrium, this current may be readily computed in terms of a magnetic potential in both the interior and exterior regions. Examples of the surface current for VMEC equilibria will be shown.

[1] Hirshman S P and Whitson J, Phys. Fluids **26** 3553 (1983)

[2] Hirshman S P, Van Rij W I and Merkel P, Comp. Phys. Comm. **43** 143–55 (1986)

[3] Merkel P, J. Comp. Phys. **66** 83–98 (1986)

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Preface - Virtual Casing Principle

- General result for an arbitrary vector field **b**

$$\mathbf{b}_V(\mathbf{r}) \equiv \frac{1}{4\pi}\int_V (\nabla'\cdot\mathbf{b})\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' + \frac{1}{4\pi}\int_V \frac{(\nabla'\times\mathbf{b})\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' =$$

$$= \frac{1}{4\pi}\oint_{\partial V} (\hat{n}'\cdot\mathbf{b})\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\times(\hat{n}'\times\mathbf{b})\,d^2\mathbf{r}' \quad + \begin{cases} \mathbf{b}(\mathbf{r}) & \mathbf{r}\in V \\ 0 & \mathbf{r}\notin V \end{cases}$$

- Details in "*The virtual-casing principle and Helmholtz's theorem*"
  - J.D. Hanson, Plasma Phys. and Control. Fusion **57** 115006 (2015)

- Previous derivation for a divergence-free field
  - "*Decoupling in the problem of tokamak plasma response to asymmetric magnetic perturbations*"
    V.D. Pustovitov, Plasma Phys. and Control. Fusion **50** 105001 (2008)

Free-Boundary VMEC fields

- Consider the plasma region D, its boundary ∂D and the coil region D^c
  - The boundary ∂D is the VMEC $s = 1$ surface.

- Label magnetic fields by the region where the current is non-zero
  - $\mathbf{B}_{plasma}$ - has nonzero curl only in D, zero curl elsewhere
  - $\mathbf{B}_{coil}$ - has nonzero curl only in D^c, zero curl elsewhere
  - $\mathbf{B}_{surf}$ - has nonzero curl only in ∂D, zero curl elsewhere. (Surface current on $s = 1$)

$\nabla\varphi$

- Total magnetic field – Biot-Savart integral over all space

$$\mathbf{B} = \mathbf{B}_{plasma} + \mathbf{B}_{coil} + \mathbf{B}_{surf}$$

$$= \frac{1}{4\pi}\int_D \frac{(\nabla'\times\mathbf{B})\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}' + \frac{1}{4\pi}\int_{D^c} \frac{(\nabla'\times\mathbf{B})\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}' + \frac{1}{4\pi}\int_{\partial D} \frac{(\nabla'\times\mathbf{B})\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^2\mathbf{r}'$$

- VMEC computes the *total* magnetic field in the plasma region

$$B_{VMEC}(\mathbf{r}) = B(\mathbf{r}) = B_{plasma} + B_{coil} + B_{surf} \qquad \mathbf{r}\in D$$

Free-Boundary VMEC fields (2)

- Define the Virtual Casing VMEC field corresponding to *any* magnetic field

$$\mathbf{B}_{any}^{VC}(\mathbf{r}) \equiv \frac{1}{4\pi}\oint_{\partial D} (\hat{n}'\cdot\mathbf{B}_{any})\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} + (\hat{n}'\times\mathbf{B}_{any})\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\, d^2\mathbf{r}'$$

- The virtual casing principle tells us that

$$\mathbf{B}_{VMEC}^{VC}(\mathbf{r}) = \mathbf{B}_{plasma} \qquad \mathbf{r}\in D^c$$

and

$$\mathbf{B}_{VMEC}^{VC}(\mathbf{r}) = \mathbf{B}_{plasma} - \mathbf{B}_{VMEC} = -\mathbf{B}_{coil} - \mathbf{B}_{surf} \qquad \mathbf{r}\in D$$

- If one is not aware of the possibility of $\mathbf{B}_{\mathrm{surf}}$, then it appears that there are two ways to compute $\mathbf{B}_{\mathrm{coil}}$ in the plasma region D:
  - A Biot-Savart integration over the external coils
  - A Virtual Casing surface integration over the plasma boundary
- Geiger et al. [Contrib. Plasma Phys. **50** 770 (2010)] add a "correction" to the VMEC field in region *D*

$$\mathbf{B}_{CORR}(\mathbf{r}) \equiv \mathbf{B}_{VMEC}(\mathbf{r}) + \left\{\mathbf{B}_{VC,VMEC}(\mathbf{r}) + \mathbf{B}_{Coil}(\mathbf{r})\right\} = \mathbf{B}_{VMEC}(\mathbf{r}) - \mathbf{B}_{surf}(\mathbf{r})$$

Free-Boundary VMEC fields (3)

- Puncture plots of this corrected field show magnetic islands in the plasma region. [Geiger et al., Poster at ISHW Padua (2013)]

- Conclusion – In some cases, there are non-trivial surface current on VMEC's plasma-vacuum interface

- How can we compute this surface current?

  Answer: Careful consideration of solutions to the interior and exterior Neumann problems

Neumann Boundary Value Problem

- Define the magnetic field with volume sources in D and D^c

$$\mathbf{B}_0 = \mathbf{B}_{plasma} + \mathbf{B}_{coil}$$

- There is no particular reason to expect that this field will be parallel to the surface of D
- Define a source-free potential Φ_{int} in the region D

$$\nabla^2 \Phi_{\text{int}} = 0 \qquad \mathbf{r} \in D \qquad\qquad (\mathbf{B}_0(\mathbf{r}) + \nabla \Phi_{\text{int}}) \cdot \hat{n} = 0 \qquad \mathbf{r} \in \partial D$$

- Green's Theorem applied to the region D becomes

$$0 = \frac{1}{4\pi} \oint_{\partial D} (-\hat{n}' \cdot \nabla' \Phi_{\text{int}}) \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \Phi_{\text{int}}\, \hat{n}' \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^2\mathbf{r}' \quad + \begin{cases} \Phi_{\text{int}} & \mathbf{r} \in D \\ \Phi_{\text{int}} / 2 & \mathbf{r} \in \partial D \\ 0 & \mathbf{r} \in D^c \end{cases}$$

Neumann Boundary Value Problem (2)

- The interior Neumann problem is then

$$\Phi_{\text{int}} + \frac{1}{2\pi}\oint_{\partial D} \Phi_{\text{int}}\, \hat{n}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^2\mathbf{r}' = \frac{1}{2\pi}\oint_{\partial D} (-\hat{n}' \cdot \mathbf{B}_0) \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} d^2\mathbf{r}' \qquad \mathbf{r} \in \partial D$$

- With some clever insight into computing the singular integral on the left-hand side [Merkel 1986 *J. Comp. Phys.* **66** 83–98], this can be solved for Φ_{int} on the plasma surface
- Then in the plasma region *D*, Green's Theorem gives

$$\Phi_{\text{int}}(\mathbf{r}) = -\frac{1}{4\pi}\oint_{\partial D} (\hat{n}' \cdot \mathbf{B}_0) \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} + \Phi_{\text{int}}\, \hat{n}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^2\mathbf{r}' \qquad \mathbf{r} \in D$$

- And in the coil region Green's Theorem gives

$$0 = \frac{1}{4\pi}\oint_{\partial D} (\hat{n}' \cdot \mathbf{B}_0) \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} + \Phi_{\text{int}}\, \hat{n}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^2\mathbf{r}' \qquad \mathbf{r} \in D^c$$

Neumann Boundary Value Problem (3)

- Similar to the internal potential, we can define an external potential

$$\nabla^2 \Phi_{ext} = 0 \qquad \mathbf{r} \in D^c \qquad\qquad (\mathbf{B}_0(\mathbf{r}) + \nabla\Phi_{ext}) \cdot (-\hat{n}) = 0 \qquad \mathbf{r} \in \partial D$$

- The exterior Neumann problem is then

$$-\Phi_{ext} + \frac{1}{2\pi}\oint_{\partial D} \Phi_{ext}\, \hat{n}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d^2\mathbf{r}' = \frac{1}{2\pi}\oint_{\partial D} (-\hat{n}' \cdot \mathbf{B}_0) \frac{1}{|\mathbf{r}-\mathbf{r}'|} d^2\mathbf{r}' \qquad \mathbf{r} \in \partial D$$

The interior (exterior) Neumann problem has a leading +(-) sign on the potential!

- Similarly, in the *coil* region

$$\Phi_{ext}(\mathbf{r}) = \frac{1}{4\pi}\oint_{\partial D} (\hat{n}' \cdot \mathbf{B}_0) \frac{1}{|\mathbf{r}-\mathbf{r}'|} + \Phi_{ext}\, \hat{n}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d^2\mathbf{r}' \qquad \mathbf{r} \in D^c$$

- And in the *plasma* region

$$0 = \frac{1}{4\pi}\oint_{\partial D} (\hat{n}' \cdot \mathbf{B}_0) \frac{1}{|\mathbf{r}-\mathbf{r}'|} + \Phi_{ext}\, \hat{n}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d^2\mathbf{r}' \qquad \mathbf{r} \in D$$

Neumann Boundary Value Problem (4)

- Combining terms from both the interior and exterior problems, the source term can be eliminated to obtain

$$\Phi_{ext}(\mathbf{r}) = \frac{1}{4\pi}\oint_{\partial D} (\Phi_{ext} - \Phi_{\text{int}})\, \hat{n}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^2\mathbf{r}' \qquad \mathbf{r} \in D^c$$

$$\Phi_{\text{int}}(\mathbf{r}) = \frac{1}{4\pi}\oint_{\partial D} (\Phi_{ext} - \Phi_{\text{int}})\, \hat{n}' \cdot \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^2\mathbf{r}' \qquad \mathbf{r} \in D$$

- Taking the gradient of these two equations, with some significant manipulation of the integrand, we can obtain

$$\mathbf{B}_{surf}(\mathbf{r}) \equiv \frac{1}{4\pi}\oint_{\partial D} (\hat{n}' \times \nabla'(\Phi_{ext} - \Phi_{\text{int}})) \times \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^2\mathbf{r}' = \begin{cases} \nabla\Phi_{\text{int}}(\mathbf{r}) & \mathbf{r} \in D \\ \nabla\Phi_{ext}(\mathbf{r}) & \mathbf{r} \in D^c \end{cases}$$

- $\mathbf{B}_{\text{surf}}$ is defined in both the coil and plasma regions
- It's definition is a Biot-Savart surface integral over the plasma surface, with surface current density $\mu_0 \mathbf{K}_{surf}(r) = \hat{n} \times \nabla(\Phi_{ext} - \Phi_{\text{int}})$

Neumann Boundary Value Problem (5)

- The magnetic field due to surface currents can be written in terms of a vector potential

$$\mathbf{B}_{surf}(\mathbf{r}) = \nabla \times \mathbf{A}_{surf}(\mathbf{r})$$

$$\mathbf{A}_{surf}(\mathbf{r}) = \frac{1}{4\pi} \oint_{\partial D} (\hat{n}' \times \nabla'(\Phi_{ext} - \Phi_{\text{int}})) \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^2\mathbf{r}'$$

- Things to note about this process
  - If the field B_0 is already tangent to the surface, the source term in the Neumann problem is zero. The solution to both the interior and exterior problems are constants, and the surface current is zero.
  - Once the exterior potential is computed, the extra work to compute the interior potential is just another linear equation solution. Most of the computational work is in setting up the surface geometry for approximating the singular surface integral.

What does VMEC compute?

- For free-boundary computations, VMEC changes the shape of the surface of D in order to ensure that $B^2/2\mu_0 + p$ is continuous across the plasma-vacuum interface.
- The vacuum field (evaluated just outside the plasma)

$$\mathbf{B}_{vac} \equiv \lim_{\varepsilon \to 0^+} \mathbf{B}(\mathbf{r} + \varepsilon \hat{n}) \qquad \mathbf{r} \in \partial D, \varepsilon > 0$$

  is decomposed as $\mathbf{B}_{vac} \equiv \mathbf{B}_{coil} + \mathbf{B}_{plasma} + \nabla\Phi_{ext}$

  - $\mathbf{B}_{\text{coil}}$ is computed from the MGRID file
  - $\mathbf{B}_{\text{plasma}}$ is approximated from the VMEC equilibrium
  - Φ_{ext} is the exterior solution to the Neumann problem, described above
  - The magnetic field just inside the plasma is the VMEC field.
  - Note – VMEC does NOT compute the interior solution to the Neumann problem

Two ways to get the surface current in VMEC

1. Modify the VMEC code to also compute the interior solution. Then

$$\mu_0 \mathbf{K}_{surf}(r) = \hat{n} \times \nabla(\Phi_{ext} - \Phi_{\text{int}})$$

1. Use the already computed vacuum field (just outside the plasma) and the VMEC field just inside the plasma to compute

$$\mu_0 \mathbf{K}_{surf}(r) = \hat{n} \times \left[\!\left[\mathbf{B} \right]\!\right] = \hat{n} \times [\lim_{\varepsilon \to 0^+} \mathbf{B}(\mathbf{r} + \varepsilon \hat{n}) - \lim_{\varepsilon \to 0^+} \mathbf{B}(\mathbf{r} - \varepsilon \hat{n})] \qquad \mathbf{r} \in \partial D$$

or

$$\mu_0 \mathbf{K}_{surf}(r) = \hat{n} \times (\mathbf{B}_{coil} + \mathbf{B}_{plasma} + \nabla \Phi_{ext} - \mathbf{B}_{VMEC})$$

evaluated appropriately.

- Will these two methods agree? How large are the surface currents?

Future Work

- At this time VMEC only computes the external potential Φ_{ext}. Modify VMEC so that it also computes the internal potential Φ_{int}.
- Modify VMEC to compute the surface current $\mathbf{K}_{\text{surf}}$ using the two method mentioned above.
- Consider as a test case a vacuum magnetic field with significant magnetic islands in the outer portion of the plasma
  - One would expect significant surface currents in this case
- Look at how $\mathbf{K}_{\text{surf}}$ varies as the outermost flux surface is moved through the island region
- Look at how $\mathbf{K}_{\text{surf}}$ varies as VMEC's radial resolution is increased.