ALGEBRA PRELIM AUGUST 14, 2013

Group Theory

Attempt at least three problems.

1. State the three Isomorphism Theorems for groups, and prove at least one.

2. Define solubility of a group in terms of its commutator subgroups, and show that a finite group G is solvable if and only if it has a subnormal series with cyclic factors.

3. Show that the following are equivalent for a finite group G:

   a) G is nilpotent.
   b) If H is a subgroup of G with $N_G(H) = H$, then $H = G$.
   c) All maximal subgroups of G are normal.
   d) All p-Sylow subgroups of G are normal.
4. State the three Sylow Theorems, and outline the proof of the first.

5. State and prove the Frattini-Argument.

Ring- and Module Theory

Attempt at least three problems.

1. Define the field of quotient of an integral domain, and show that it always exists.

2. State the fundamental theorem for finitely generated modules over a PID. Use it to determine the number (up to isomorphism) of Abelian groups of order 3500. Give s complete list of the isomorphism classes.

3. Give an example of a UFD which is not a PID, and prove your claim.

4. Let R be a ring.
   (a) State the Artin-Wedderburn Theorem for R.
   (b) Show that $Mat_{2x2}(D)$ is simple (i.e. has no proper 2-sided ideals) if D is a division ring.
   (c) Give an example of a division ring D that is not a field.

5. Let R be an integral domain.
   (a) Give a definition for R being a Dedekind domain.
   (b) Provide an example of a Dedekind domain that is not a PID.

Fields

Attempt at least three problems.

1. Find the Galois group of $f(x) = x^4 + 2$ over $\mathbb{Q}$.

2. Show that a field extension $E \supseteq K$ is algebraic if and only if every subring S of E which contains K is a subfield.

3. Let F be a field extension of a field K.
   (a) Give a definition for F being a Galois extension of K.
   (b) State the Fundamental Theorem of Galois for the extension F over the field K.

4. Let F be the splitting field of $x^3 - 2$ over $\mathbb{Q}$.
   (a) Determine the Galois group of F as an extension over $\mathbb{Q}$.
   (b) Determine all subfields between F and $\mathbb{Q}$.

5. Show that an irreducible polynomial over a field of characteristic 0 cannot have multiple roots.