Matrices PreLim Exam
Summer 2009

Answer all of the *State* type problems, work all of the numerical examples, and 5 of the *Prove* type problems.

(1) State the spectral mapping theorem. Prove the same.
(2) State Gershgorin's Theorem. Prove the same.
(3) Let $A \in \mathbf{Q^{4\times 4}}$. Suppose that as a complex matrix, A is similar to

$$B = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Find the Smith Normal Form of $A - tI \in \mathbf{Q[t]^{4\times 4}}$. Repeat for

$$B = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(4) Find 3×3 symmetric matrix P such that for all $v \in R^3$, Pv is the point on the plane $x - y + z = 0$ closest to v.
(5) State Schur's Triangularization theorem. Outline a proof of the same.
(6) State the definition of normal matrix, and unitary matrix. Prove that a matrix is normal if and only if it is unitarily diagonalizable.
(7) Hermitian Matrix Eigenvalues. Let A be $n \times n$ complex Hermitian.
   (a) State the interlacing inequalities.
   (b) State the inequalities that exist between the eigenvalues of A and the diagonal entries of A
   (c) Discuss the converses of the theorems stated in part (a) and (b).
   (d) Prove that (a) implies (b).
(8) State Perron's Theorem for positive matrices.
(9) Nonnegative irreducible matrices
   (a) State the definition of irreducible matrix.
   (b) State the definition and or characterization of primitive matrix.
   (c) Suppose that A is nonnegative and irreducible with $k > 1$ eigenvalues of maximum absolute value. State some of equivalent characterizations of the number k
   (d) Use part (c) to prove that a symmetric nonnegative irreducible matrix is either primitive or permutation similar to this block form:

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

(10) Norms
   (a) State the definition of *norm* $|| \cdot ||$on $\mathbf{R^n}$.
   (b) State the definition of the operator norm induced on $\mathbf{R^{n\times n}}$ by $|| \cdot ||$.
   (c) Prove that the induced norm satisfies the inequality $||AB|| \leq (||A||)(||B||)$